STABILITY ANALYSIS OF EXTENDED GYROSTATS

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Approved:

D. J. McGill, Chairman

A. W. Marris

W. W. King

M. Stallybrass

Date approved by Chairman 5/31/76

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SUMMARY

This thesis presented a mathematical analysis for the stability of spinning rigid bodies. First, the stability of a general unsymmetrical rigid body with a fixed point was considered. Floquet's theory, Routh's solution, and Rumiantsev's analysis were applied to the problem to classify all regions within the limits of possible rigid bodies for the chosen parameter plane regarding their stability. Second, the same problem was extended to that of a gyrostat with a fixed point in which the effects of relative spin on the stability-instability regions of the previous problem were examined. Here, a Routh-type analysis of the linearized equations and Rumiantsev's Lyapunov analysis were employed to yield stability-instability regions for the range of possible rigid bodies. Next, lateral inertia imbalance was introduced to the 'rotor' of the gyrostat giving rise to a system we named an 'extended' gyrostat. For this more complicated problem, the equations of motion were linearized and observed to be of the form required by Floquet's theory for a stability analysis. Thus, with an introduction of an inertia imbalance ratio parameter for the rotor and an axial inertia ratio parameter for the ratio between the axial inertia of the rotor and the main body with the fixed point, a Floquet analysis was conducted for several cases and a comparison to the results obtained for the gyrostat were made. An unsuccessful attempt to employ the direct method of Lyapunov to the stability of the extended gyrostat was discussed. Finally, the problem of the attitude stability of the unsymmetrical satellite with an unsymmetrical rotor in a circular

orbit about the earth was discussed. The equations of motion for this problem were derived, and mathematical difficulties arising in a stability analysis were mentioned.

Presently the author and his advisor are extending the work of this thesis to include the effects of viscous damping on the stability regions.

The conclusions reached in this thesis are:

1. For the dynamical systems considered in this work, the stability-instability regions of a single rigid body seem to be strongly affected when relative motion is introduced into the system by means of a second axisymmetric body to form a gyrostat. It was observed that both beneficial and harmful effects can be produced. With the introduction of relative spin, a new but small infinitesimally stable region appeared in the region of possible rigid bodies within the parameter plane.

2. In the first case investigated, Floquet's theory, Routh's analysis of the linearized equations, and Rumiantsev's Lyapunov analysis were shown to complement each other very well.

3. For the investigation of the extended gyrostat, it was observed that for an axial inertia ratio between the rotor and main body of $\bar{c} = 0.1$ the Floquet results seem to be almost identical to that of a gyrostat. But when $\bar{c} = 0.3$ the effect of lateral inertia imbalance becomes more noticeable. Of greatest importance was the effect upon the Lyapunov stable region which was observed to contain more and more unstable points as the inertia imbalance parameter $\bar{e}$ was increased.

A list of the accomplishments of this thesis along with suggested topics for future investigation are given in Tables 1 to 3.

Table 1. Accomplishments for the Problem of an Unsymmetrical Body with a Fixed Point.

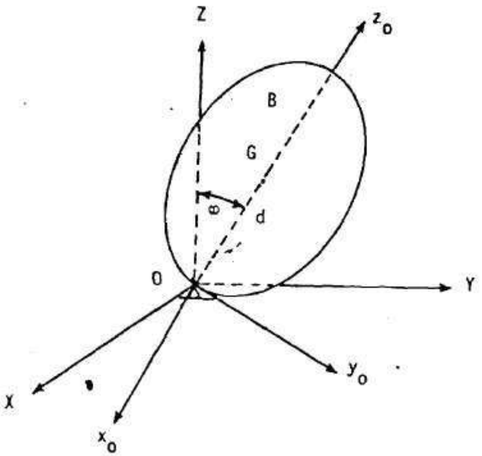
Problem 1	Accomplishments	Possible Future Investigations
Unsymmetrical Rigid Body with a Fixed Point.  Parameters: $i_x = I_{x_o}/I_{z_o}$, $k = I_{y_o}/I_{x_o}$, $S = \frac{I_{z_o}\omega_{z_i}^2}{mgd}$ Shown is the unsymmetrical top; for negative S the Problem is called a suspended gyro; both problems were investigated.	1. Floquet's Theory was used to obtain stability and instability conditions for the linearized equations, the latter of which are also valid for the complete nonlinear system. 2. The stability-instability conditions of the linearized equations that were obtained by Routh [4] were corrected and verified by the Floquet analysis. 3. Rumiantsev's Lyapunov analysis was utilized to yield sufficient conditions for stability of the full nonlinear system.	1. Regions that could only be shown to be infinitesimally stable are open to investigation with regard to stability of the full nonlinear system. 2. An examination of the effects of damping on the stability regions might be considered. 3. Other equilibrium positions might be sought and examined for stability.

Table 2. Accomplishments for the Problem of an Gyrostat with a Fixed Point.

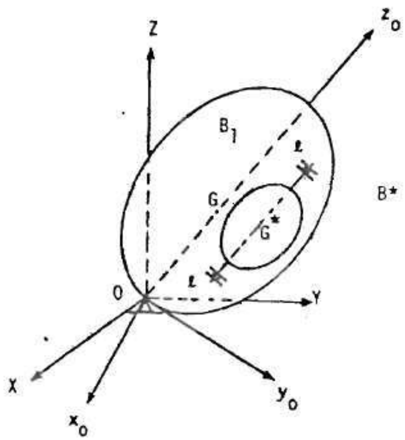
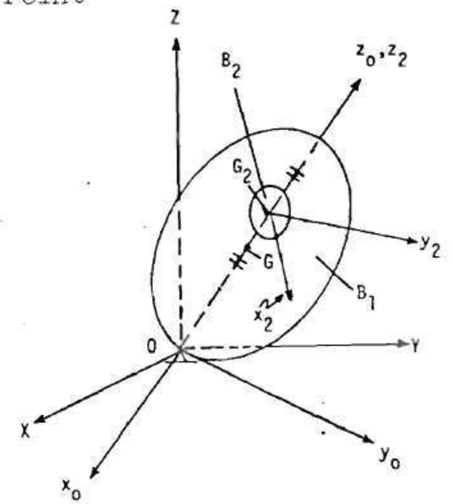
Problem 2	Accomplishments	Possible Future Investigations
Gyrostat with a Fixed Point	1. Stability-instability conditions for the linearized equations were obtained via an analysis equivalent to that used by Routh in investigating the stability of a spinning unsymmetrical top.	1. The zones that could only be shown to be infinitesimally stable remain to be investigated regarding stability of the full nonlinear system.
	2. Rumiantsev's Lyapunov analysis for this problem yielded sufficient conditions for stability of the full nonlinear system.	2. The effects of damping on both the stable and infinitesimally stable regions could be investigated.
Parameters: i_x, s, k, β β = nondimensional relative spin speed (Problem also worked for suspended gyrostat, having G below O)	3. It was shown that the addition of relative spin could cause both beneficial and harmful effects upon the stability of a single spinning unsymmetrical body.	3. Equilibrium positions including various rotor orientations might be sought and investigated for stability.

Table 3. Accomplishments for the Problem of an Extended Gyrostat with a Fixed Point.

Problem 3	Accomplishments	Possible Future Investigations
Extended Gyrostat with a Fixed Point  Parameters: i_x, s, k, β, $\bar{c}$, $\bar{e}$ $\bar{c}$ = axial inertia ratio $\bar{e}$ = nondimensional inertia imbalance of rotor. (Problem also worked for mass center of the extended gyrostat below the fixed point.)	1. The equations of motion were derived and an equilibrium position was obtained. 2. A Lyapunov analysis of the full nonlinear system was attempted and the difficulties arising were explained. 3. The equations of motion were linearized yielding a set of first order differential equations with periodic coefficients. 4. Floquet's theory was utilized to obtain instability and infinitesimally stable regions. 5. The effect of inertia imbalance on the stability regions was obtained, as well as the effect of relative z_o-axis inertia of the rotor vs. the main body.	1. Only infinitesimal stability could be shown; hence, any conditions for which stability of the full nonlinear system would be a new result. 2. The effects of damping might also be investigated. 3. A stability analysis in which other rotor positions are considered is open for investigation.

A thesis of this type would not be complete if it did not offer some practical advice to potential future stability investigators. These recommendations are:

1. Carefully derive the equations of motion for the problem under investigation, making sure the equations are of the simplest form possible. Equilibrium solutions should then be noted.

2. If integrals of motion can be easily obtained then a Lyapunov analysis is suggested. Here, both stability and instability theorems should be employed if applicable.

3. If a Lyapunov-type analysis does not yield both necessary and sufficient conditions for stability or fails altogether, then a linearized approach should be taken which would at least yield sufficient conditions for instability.

4. If the coefficients of the linearized equations are periodic and continuous in time, then Floquet's theory can be employed as was done in this thesis.

CHAPTER I

INTRODUCTION

Stability problems of spinning rigid bodies have received extensive attention in the past two decades, due mainly to an increasing interest in spaceflight. Numerous investigations of such problems have been made and can easily be found in the literature. Considerable credit should be given to the Russian dynamicist Rumiantsev [11][1], who in 1956 derived sufficient conditions for the stability of a rigid body spinning about a principal axis through a fixed point of the body, with no requirement that the other two principal moments of inertia with respect to the fixed point be equal. His results have been extended to include, for example, Newtonian force fields [14], arbitrary potential fields of force [13], the Euler case [9], and other stability problems related to the rigid body under a uniform field of force. In many of these investigations, Rumiantsev and his followers made extensive use of the methods developed by A. M. Lyapunov [1] in 1892 for determining the stability of systems of differential equations. E. J. Routh [4], an English dynamicist and mathematician, considered the general stability problem of a heavy unsymmetrical spinning top, using a linear analysis to examine its stability. Unfortunately, only the instability results of Routh's analysis were extendible to the complete nonlinear system. This fact follows from

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[1] Numbers in brackets refer to references listed in Literature Cited.

Lyapunov's theorem on the stability in the first approximation.[1]

We define a gyrostat G to be a mechanical system consisting of a rigid body B_1 and other bodies $B_2, \ldots, B_n$, either rigid or not, whose relative motions cannot alter the mass geometry of the system. Thus for a gyrostat, the mass center is fixed in G and the inertia properties do not change with time.

Examples of such a system are: a solid body to which there are connected axes of several symmetric rotors; or a solid body with a cavity of arbitrary shape entirely filled with a homogeneous incompressible fluid.

Gyrostats are frequently encountered in the field of space flight, when rotors are employed to enhance the attitude stability of a space vehicle. Also, a torpedo utilizes a gyrostat to maintain the direction in which it is started.

Using the direct method of Lyapunov, Rumiantsev [17] investigated the stability of certain motions of heavy[2] gyrostats with a fixed point. In the case where the mass center of the gyrostat is taken to be the fixed point, he obtained sufficient conditions for both stability and instability of the permanent rotations. The same problem was independently considered by Kane and Fowler [19] using a different approach, and was also solved in a Ph.D. thesis by Crespo da Silva (published as [20]). The results of all three investigations were equivalent. Several other cases were also examined by Rumiantsev

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[1] See ref. [7], page 227.

[2] The term "heavy" always refers to the case of bodies under uniform gravity in the literature on gyroscopes and gyrostats.

[17,18], Anchev [21,22], Kolensnikov [23], and others.

In the design of spacecraft attitude control systems, all torques that tend to disturb the attitude of a spacecraft must be considered. One of these torques is the gravitational or gravity gradient torque which results from the variation in the gravitational force over the distributed mass of the spacecraft. Expressions for such torques have been obtained by Roberson [29,30], Nidey [31], Hultquist [32], and Lur'e [33].

Since many of the first satellites were spin stabilized, the problem of predicting the motion of the spin axis due to gravitational disturbance torques has been extensively investigated. One of the first investigations of such satellites was made by Thomson [36], who considered the case of an axisymmetric satellite in a circular orbit about the earth. For this case, he investigated the effect of spin about the axis of symmetry upon the attitude stability of the satellite. However, his work was in error as detected and corrected by Kane, Marsh, and Wilson [37].

The studies by Thomson and Kane, Marsh, and Wilson were restricted in scope to attitudes in which the symmetry axis is normal to the orbital plane, and were confined to analysis by linearization. Likins [42] was able to show that there exist other attitudes in which the symmetry axis remains stationary in an orbiting reference frame. By using Lyapunov's second method he was about to obtain stability regions that were only known to be infinitesimally stable prior to his investigation.

Using Floquet's theory, Kane and Shippy [39] investigated the attitude stability of a spinning unsymmetrical satellite in a circular orbit. Their investigation clearly exemplified the effects of inertial eccentricity on the attitude stability of the satellite.

In a later investigation, Kane and Mingori [40] extended the work of Kane and Shippy by examining the effects of a rotor on the attitude stability of the satellite in a circular orbit. They were able to show that both beneficial and harmful effects could be produced rather easily by low-speed rotors.

Let us now define an "extended gyrostat" G^* to be two or more coupled rigid bodies (called $B_1,\ldots,B_n$) with B_1 being called the main body and having the following properties:

(i) like a gyrostat, the mass center of G^* remains fixed with time in B_1.

(ii) unlike a gyrostat, the inertia properties of G^*, written with respect to a point of B_1, may vary with time.

An example of an extended gyrostat is illustrated in Figure 1.

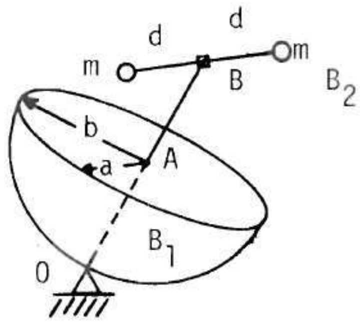

Figure 1. An Example of an Extended Gyrostat.

The system of Figure 1 is composed of two rigid bodies B_1 (a semi-ellipsoidal top) and B_2 (a massless rod with a mass m concentrated at B) and B_2 is free to rotate about $\overline{AB}$ only in a manner in which it always remains perpendicular at $\overline{AB}$.

Note that if either $a = b$ or if two more masses m are added to B_2 to form a symmetrical cross, then the system will become a gyrostat.

This dissertation presents stability investigations of spinning rigid bodies. First, the classical problem of a single arbitrary rigid body spinning about an axis which passes through a fixed point was investigated for stability. In this investigation, all possible regions regarding moments of inertia and spin speed were clearly defined as to its stability. In order to study the effects of relative spin on stability, the same analysis was repeated for the gyrostat with a fixed point. Finally, the stability of the extended gyrostat was investigated to add the effects of lateral inertial imbalance.

In the above investigations, Floquet theory, Lyapunov's second method (the direct method of Lyapunov), and the procedure followed by Routh were used whenever feasible.

CHAPTER II

STABILITY REGIONS FOR AN UNSYMMETRICAL RIGID BODY WITH A FIXED POINT

2.1 Derivation of the Equations of Motion

Let us consider a rigid body B with a point O which is fixed in an inertial reference frame F constituted by axes (X, Y, Z). In Figure 2, the orthogonal axes (x_o, y_o, z_o) are permanently fixed in the rigid body B, and they are principal axes of B for point O. The mass center G of B lies on the z_o-axis. We denote the z_o-coordinate of G by the letter d, which represents the signed distance from O to G.

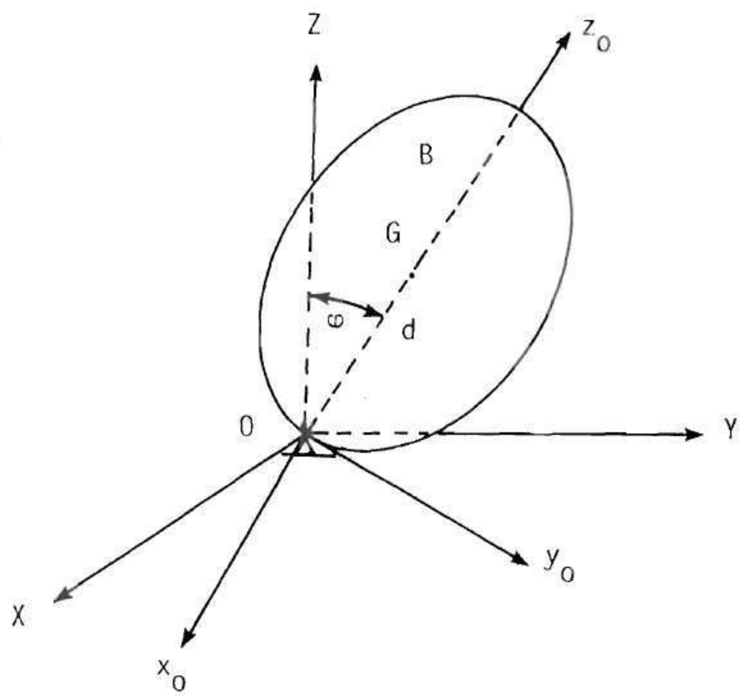

Figure 2. Inertial and Body-fixed Frames of Spinning Rigid Body.

If we align (x_o, y_o, z_o) with (X,Y,Z), then perform successive rotations of the body through angles θ_1, θ_2, and θ_3 about the respective current positions of x_o, y_o, and z_o, we may write the angular velocity vector of the body B in F as

$$\vec{\omega} = (C_2C_3\dot{\theta}_1 + S_3\dot{\theta}_2)\hat{i} + (C_3\dot{\theta}_2 - C_2S_3\dot{\theta}_1)\hat{j} + (\dot{\theta}_3 + S_2\dot{\theta}_1)\hat{k} \ , \qquad (2.1)$$

in which $(\hat{i},\hat{j},\hat{k})$ are unit vectors directed in the positive sense along the respective body axes (x_o, y_o, z_o) of B. Also $(\dot{\ }) = \frac{d}{dt}$ denotes differentiation with respect to time. Notationally $S_1 = \sin\theta_1$, $C_2 = \cos\theta_2$, etc.

Since the body axes (x_o, y_o, z_o) are principal, the angular momentum of B with respect to its fixed point O is clearly

$$\vec{H}_o = I_{x_o}\omega_x\,\hat{i} + I_{y_o}\omega_y\,\hat{j} + I_{z_o}\omega_z\,\hat{k} \ , \qquad (2.2)$$

in which $(I_{x_o}, I_{y_o}, I_{z_o})$ are the principal moments of inertia of B for O, with respect to axes (x_o, y_o, z_o), and $(\omega_x,\omega_y,\omega_z)$ are the corresponding angular velocity components in Equation (2.1). Since O is permanently fixed in F, the Euler equations for B follow from the vector equation

$$\overset{F\cdot}{\vec{H}_o} = \overset{B\cdot}{\vec{H}_o} + \vec{\omega} \times \vec{H}_o = \vec{M}_o \ , \qquad (2.3)$$

where $\overset{F\cdot}{\vec{H}_o}$ and $\overset{B\cdot}{\vec{H}_o}$ indicate the time derivatives of $\vec{H}_o$ as observed in the F and B frames respectively, and $\vec{M}_o$ is the moment about O.

Assuming that the only moment acting on B about O is that due to uniform gravity, we see that

$$\vec{M}_o = d\hat{k} \times (-mg\,\hat{K}) \quad , \tag{2.4}$$

where $\hat{K}$ is an upward vertical unit vector, m is the mass of B, and g is the gravitational acceleration, assumed uniform.

Expressing $\hat{K}$ in terms of $(\hat{i},\hat{j},\hat{k})$ and substituting Equations (2.1-2.4) into Equation (2.3) yields the Euler equations for the present problem:

$$I_{x_o}\dot{\omega}_x - (I_{y_o} - I_{z_o})\omega_y\omega_z = mgd(S_1C_3 + S_2S_3C_1) \quad , \tag{2.5}$$

$$I_{y_o}\dot{\omega}_y - (I_{z_o} - I_{x_o})\omega_z\omega_x = mgd(S_2C_1C_3 - S_1S_3) \quad , \tag{2.6}$$

$$I_{z_o}\dot{\omega}_z - (I_{x_o} - I_{y_o})\omega_x\omega_y = 0 \ . \tag{2.7}$$

2.2 Stability Analysis Via Floquet's Theory

An equilibrium solution of Equations (2.5-7) is clearly

$$\theta_1 = \theta_2 = 0 \quad , \tag{2.8a}$$

$$\theta_3 = \omega_{z_i}\ t \ , \tag{2.8b}$$

with ω_{z_i} representing a constant undisturbed angular spin speed of B about its z_o-body axis. It is also clear that this equilibrium solution corresponds to $(\omega_x,\omega_y,\omega_z) = (0,0,\omega_{z_i})$, which follows easily from Equation (2.1).

In this stability analysis, we are concerned as to whether or not the angle φ between the verticle z-axis and the body fixed z_o-axis grows with time. From the definition of dot product of vectors it is obvious that

$$\cos \varphi = \hat{K} \cdot \hat{k} , \tag{2.9}$$

which yields after some calculations,

$$\varphi = \cos^{-1} (\cos \theta_1 \cos \theta_2) \quad . \tag{2.10}$$

Thus, it is clear that the tilt angle φ depends only on angles θ_1 and θ_2 but not on angle θ_3.

To examine the stability of the present motion, we need only disturb it by setting

$$\theta_1 = \epsilon_1 , \tag{2.11}$$

$$\theta_2 = \epsilon_2 , \tag{2.12}$$

in which ϵ_1 and ϵ_2 are small functions of time, perturbations of the undisturbed θ_i, $i = 1,2$. These new angles and their derivatives must of course satisfy Equations (2.5-2.7).

After linearization in the perturbations, Equation (2.7) gives zero equals zero. Hence for the present problem, the stability of B is governed only by the solutions of Equations (2.5,2.6) if linearized in ϵ_1 and ϵ_2.

Substituting Equations (2.8b, 11, 12) into Equation (2.1) and the reulsts for ω_x, ω_y, and ω_z into Equations (2.5,2.6) yields a pair of second order differential equations which may be brought to first order by the introduction of functions δ_1 and δ_2:

$$\delta_1 = \epsilon_1' = \frac{\dot{\epsilon}_1}{\omega_{z_i}} \tag{2.13}$$

$$\delta_2 = \epsilon_2' = \frac{\dot{\epsilon}_2}{\omega_{z_i}} \ , \tag{2.14}$$

in which primes indicate differentiation with respect to the nondimensional time τ given by

$$\tau = \omega_{z_i} t \quad . \tag{2.15}$$

Thus δ_1 and δ_2 represent the rates of growth with nondimensional time of the tipping disturbance angles ϵ_1 and ϵ_2.

Now, let us define two nondimensional inertia parameters i_x and k and a nondimensional spin parameter S as follows:

$$i_x = I_{x_o}/I_{z_o} \quad , \tag{2.16}$$

$$k = I_{y_o}/I_{x_o} \quad , \tag{2.17}$$

$$S = I_{z_o}\omega_{z_i}^2/(mgd) \quad . \tag{2.18}$$

Using these quantities, we may solve for δ_1' and δ_2' from Equations (2.5) and (2.6), yielding after linearization

$$\delta_1' = \left[\frac{S_3^2 + k\,C_3^2}{k\,i_x\,S}\right]\epsilon_1 + \left[\frac{(k-1)S_3C_3}{k\,i_x\,S}\right]\epsilon_2 + \left[\frac{S_3C_3[k-1 + i_x(1-k^2)]}{k\,i_x}\right]\delta_1 + \left[\frac{S_3^2(1-i_x) + k\,C_3^2(i-ki_x) + ki_x}{k\,i_x}\right]\delta_2 \ , \tag{2.19}$$

$$\delta_2' = \left[\frac{S_3C_3(k-1)}{ki_xS}\right]\epsilon_1 + \left[\frac{C_3^2 + k\,S_3^2}{ki_xS}\right]\epsilon_2 + \left[\frac{S_3^2(1-ki_x)k + C_3^2(1-i_x) + ki_x}{ki_x}\right]\delta_1$$

$$+ \left[\frac{-S_3C_3[k-1 + i_x(1-k^2)]}{ki_x}\right]\delta_2 \ . \qquad (2.20)$$

Equations (2.13, 2.14, 2.19, 2.20) govern the stability of the linearized system of the present problem. Note that it is physically obvious that overall stability results cannot be affected by swapping the moments of inertia I_{x_o} and I_{y_o}. According to Equations (2.16, 2.17), this may be effected mathematically by replacing i_x by ki_x and k by 1/k. Thus, all stability results that can be obtained for $k \epsilon [1,\infty)$[1] are identical with those obtainable for the reciprocal values, i.e. for $k \epsilon (0,1]$. We may thus, with no loss in generality, restrict our analysis to values of k from zero to unity. Also, from the definition of moment of inertia, it follows immediately that the sum of any two of the principal moments of inertia is always greater than or equal to the third. In terms of the present parameters, this means that

$$\frac{1}{1+k} \le i_x \le \frac{1}{1-k} \ , \qquad (2.21)$$

where $k \epsilon (0,1]$.

Hence, for a given value of k between 0 and 1, inclusive, we may use S as ordinate and i_x as abscissa, and with no loss in generality, we need examine stability only between the limits given by the inequalities of Equation (2.21).

---

[1]The notation $k \epsilon [1,\infty)$ means k satisfies the inequality $1 \le k < \infty$.

Equations (2.13, 2.14, 2.19, 2.20) may be expressed in matrix form as

$$\{X\}' = [A(\tau)]\{X\} , \tag{2.22}$$

in which

$$\{X\} = \text{col. } [\epsilon_1, \epsilon_2, \delta_1, \delta_2] \tag{2.23}$$

and it is clear that $[A(\tau)]$ is a periodic matrix of period π.

The stability of such a periodic system may be determined by Floquet's theory (see Reference [8], pages 55-58). To use this theory, we integrate the matrix equation

$$[H(\tau)]' = [A(\tau)]\ [H(\tau)] \tag{2.24}$$

from $\tau = 0$ to the period $\tau = \pi$ with initial condition $[H(0)] = [I]$ (the 4×4 unit matrix), and examine the moduli $|\lambda_i|$ of the four eigenvalues λ_i of $[H(\pi)]$. Then in accordance with Floquet's theory, all solutions $\{X\}$ (i.e., all sets $(\epsilon_1, \epsilon_2, \delta_1, \delta_2)$ satisfying Equation (2.22)) are bounded as $\tau \to \infty$ if and only if $|\lambda_i| \leq 1$, $i = 1,2,3,4$, and where for those λ_i for which $|\lambda_i| = 1$ the multiplicity μ_i of λ_i equals the nullity ν_i 9f the matrix $[H(\tau)] - \lambda_i[I]$. The system has a periodic solution if and only if there is at least one eigenvalue $\lambda_i = 1$.

It is important to note that if an instability result for the linear system (Equation (2.22)) is indicated by at least one $|\lambda_i| > 1$ such a result is extendible to the full nonlinear system. However, while all four $|\lambda_i| \leq |$ will correctly indicate stability for the linearized system, this result is unfortunately not guaranteed to be true also for the nonlinear system.

Figure 3 presents the stability-instability regions obtained using Floquet's theory for a typical inertia imbalance ratio of $k = 0.7$. Here regions b and e were found to be unstable, while regions a, c, d, and f were observed to be at least infinitesimally stable. Of course, only isolated points (i_x, S) within and on the boundaries of these regions could be examined, but it is felt that enough such points were checked within each region to be reasonably sure of instability therein.

2.3 Determination of Stability Regions Via the Direct Method of Lyapunov

Following closely the analysis of Rumiantsev [11], it is possible for us to obtain, for the present problem, sufficient conditions for stability of the full nonlinear system.

Here we let $(\gamma_1, \gamma_2, \gamma_3)$ represent the direction cosines of the upward vertical referred to the respective body axes (x_o, y_o, z_o). Thus, we can write $\hat{K}$ in terms of γ_1, γ_2, and γ_3:

$$\hat{K} = \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k} \quad . \tag{2.25}$$

Now, differentiation of $\hat{K}$ in the inertial frame gives rise to the classical Poisson kinematical equations:

$$\dot{\gamma}_1 = \omega_z \gamma_2 - \omega_y \gamma_3 \quad , \tag{2.26}$$

$$\dot{\gamma}_2 = \omega_x \gamma_3 - \omega_z \gamma_1 \quad , \tag{2.27}$$

$$\dot{\gamma}_3 = \omega_y \gamma_1 - \omega_x \gamma_2 \quad . \tag{2.28}$$

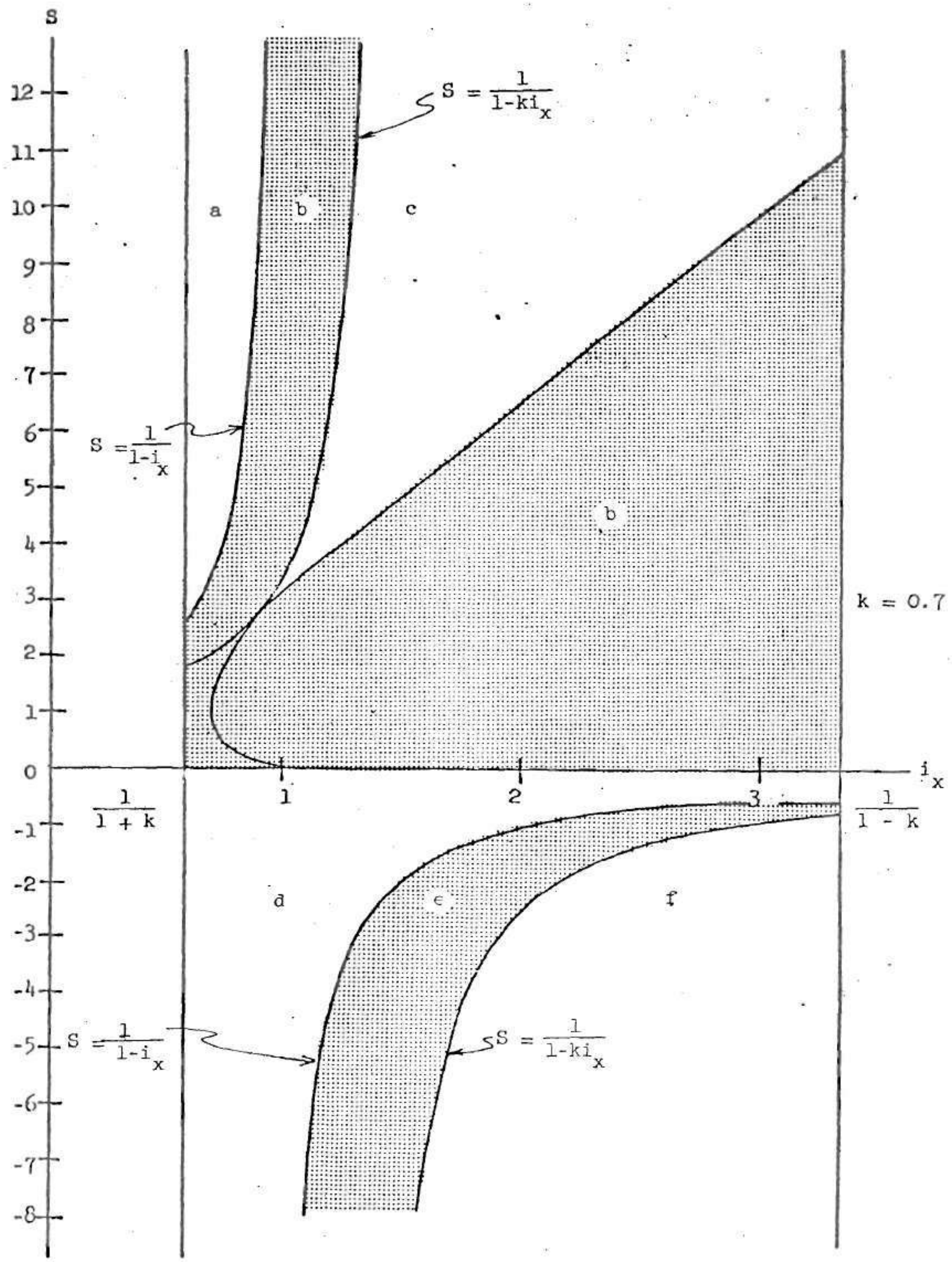

Figure 3. Stability and Instability Regions for a Typical Inertia Imbalance Ratio, $k = 0.7$.

Rewriting the right hand side of each of the Equations (2.6-2.8) in terms of direction cosines, we obtain:

$$I_{x_o}\dot{\omega}_x - (I_{y_o} - I_{z_o})\omega_y\omega_z = mgd\ \gamma_2 \quad , \tag{2.29}$$

$$I_{y_o}\dot{\omega}_y - (I_{z_o} - I_{x_o})\omega_z\omega_x = -\ mgd\ \gamma_1 \quad , \tag{2.30}$$

$$I_{z_o}\dot{\omega}_z - (I_{x_o} - I_{y_o})\omega_x\omega_y - 0 \quad . \tag{2.31}$$

Let us multiply Equations (2.29-2.31) by ω_x, ω_y, and ω_z respectively, and add the results. Then, in view of Equation (2.28), we obtain the first integral

$$I_{x_o}\omega_x^2 + I_{y_o}\omega_y^2 + I_{z_o}\omega_2^2 + 2mgd\ \gamma_3 = \text{const.}, \tag{2.32}$$

which is referred to as the energy integral.

Next, let us multiply Equations (2.29-2.31) by γ_1, γ_2, and γ_3, respectively, and add the result. Then by Equations (2.26-2.28), we obtain the second (momentum) integral

$$I_{x_o}\omega_x\ \gamma_1 + I_{y_o}\omega_y\ \gamma_2 + I_{z_o}\omega_2\gamma_3 = \text{const.} \tag{2.33}$$

Obviously, Equations (2.26-2.28) admit the geometric integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \quad . \tag{2.34}$$

Equations (2.32-2.34) are referred to in the literature as the integrals of motion for a rigid body with a fixed point. It is noteworthy to mention that in special cases additional integrals might be obtained, for example the cases of Euler [9], Kovalevskaya [10], and Lagrange [5].

Clearly, $(\gamma_1, \gamma_2, \gamma_3) = (0,0,1)$ and $(\omega_x, \omega_y, \omega_z) = (0,0,\omega_{z_i})$ where ω_{z_i} = const., is an unperturbed solution to Equations (2.26-2.31). Thus, allowing small perturbations $(\omega_x, \omega_y, \omega_z) = (\xi_1, \xi_2, \omega_2 + \xi_3)$ and $(\gamma_1, \gamma_2, \gamma_3) = (\eta_1, \eta_2, 1 + \eta_3)$ in the angular velocity components and the direction cosines and substituting these quantities into Equations (2.32-2.34) we obtain the following integrals:

(a) Conservation of energy:

$$V_1 = I_{x_o}\xi_1^2 + I_{y_o}\xi_2^2 + I_{z_o}(\xi_3^2 + 2\omega_{z_i}\xi_3) + 2mgd\,\eta_3 = \text{const.}, \qquad (2.35)$$

(b) Conservation of angular momentum about the vertical:

$$V_2 = I_{x_o}\xi_1\eta_1 + I_{y_o}\xi_2\eta_2 + I_{z_o}(\omega_{z_i}\eta_3 + \xi_3 + \xi_3\eta_3) = \text{const.} \qquad (2.36)$$

(c) Sum of squares of γ_i add to unity:

$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2\eta_3 = 0 \quad . \qquad (2.37)$$

Following the ingenious work of Chetayev [5], Rumiantsev constructed the following Lyapunov V-function:

$$V = V_1 - 2\omega_{z_i}V_2 + (I_{z_o}\omega_{z_i}^2 - mgd)V_3 + \tfrac{1}{4}\mu\, V_3^2 \;, \qquad (2.38)$$

where μ is an arbitrary constant. With this choice of V, the linear terms cancel. Also, $\dot{V}$ is trivially negative semidefinite, since $\dot{V} \equiv 0$.

A discussion of the direct method of Lyapunov is presented in Appendix C.

Substituting Equations (2.35-2.37) into Equation (2.38) yields

$$V = I_{x_o}\xi_1^2 + I_{y_o}\xi_2^2 + I_{z_o}\xi_3^2 - 2\omega_{z_i}(I_{x_o}\xi_1\eta_1 + I_{y_o}\xi_2\eta_2 - I_{z_o}\xi_3\eta_3)$$

$$+ (I_{z_o}\omega_{z_i}^2 - mgd)(\eta_1^2 + \eta_2^2 + \eta_3^2) + \mu\eta_3^2 + \frac{\mu}{4} f(\eta_1,\eta_2,\eta_3) , \quad (2.39)$$

in which

$$f = (\eta_1^2 + \eta_2^2 + \eta_3^2)(\eta_1^2 + \eta_2^2 + \eta_3^2 + 4\eta_3) \quad . \quad (2.40)$$

The function V will be positive definite with respect to the variables ξ_1, ξ_2, ξ_3, η_1, η_2 and η_3 if its quadratic part, i.e. $V - \frac{\mu}{4} f$, is positive definite. Necessary and sufficient conditions for a quadratic function to be positive definite can be obtained by Sylvester's criterion. For the present case, six inequalities are obtained. Three of these trivially require that I_{x_o}, I_{y_o}, and I_{z_o} be positive. The other three inequalities are:

$$I_{z_o}\omega_{z_i}^2 - mgd + \mu > 0 , \quad (2.41)$$

$$S > \frac{1}{1-ki_x} \quad \text{for} \quad i_x < 1/k \quad \text{and} \quad S < \frac{1}{1-ki_x} \quad \text{for} \quad i_x > 1/k , \quad (2.42)$$

$$S > \frac{1}{1-i_x} \quad \text{for} \quad i_x < 1 \quad \text{and} \quad S < \frac{1}{1-i_x} \quad \text{for} \quad i_x > 1 , \quad (2.43)$$

where we take

$$\mu = mgd \quad .$$

Satisfaction of these conditions guarantee Lyapunov stability for the full nonlinear system in regions (a) and (d) of Figure 3. However, Rumiantsev's Lyapunov function fails to give us information regarding the stability of the remaining regions.

2.4 Routh's Solution

Another approach to the present problem was considered by Routh [4], in which he linearized the Euler and Poisson equations in both direction cosines and angular velocity components obtaining four first-order linear differential equations. Setting $\gamma_1 = A_1 e^{i\lambda t}$, $\gamma_2 = A_2 e^{i\lambda t}$, $\omega_x = A_3 e^{i\lambda t}$, and $\omega_y = A_4 e^{i\lambda t}$, he obtained a set of four homogeneous linear algebraic equations in the Ai which gave rise to a characteristic equation quadratic in λ^2. To ensure that both roots λ^2 were real and positive, three inequalities were obtained from the characteristic equation. In the present notation these inequalities are:

$$S^2[i_x(1+k) - 1]^2 + 2i_x S\{[i_x(1+k) - 4ki_x^2 - 1](1+k) + 4ki_x\}$$
$$+ i_x^2(1-k)^2 > 0 \tag{2.44}$$

(ensures two real roots),

$$[(1-i_x)S - 1][(1-ki_x)S - 1] > 0 \tag{2.45}$$

(ensures the roots have the same sign),

$$[S(1 + 2ki_x^2 - i_x - ki_x) - i_x(1+k)] \cdot \mathrm{sgn}(d) > 0 \tag{2.46}$$

(ensures the roots are both positive),

where

$$\mathrm{sgn}(d) = d/|d| \ .$$

Satisfaction of all of these inequalities by sets (k, i_x, s) ensures that the direction cosines and angular velocity components remain bounded, and hence that the system is infinitesimally stable. Failure to satisfy any one of these inequalities results in instability.

It can easily be shown that for $S < 0$ (i.e. $d < 0$) conditions (2.44, 2.46) are trivially satisfied for all values of i_x within the range of possible rigid bodies. Thus in the case of negative S, the stability of the linearized system is governed only by condition (2.45).

In Routh's solution [4], the third inequality was in error[1], and has been corrected here. With these corrections, the curves and corresponding zones defined by conditions (2.44-2.46) are depicted in Figure 3 for the typical inertia imbalance ratio of $k = 0.7$.

2.5 General Comments

The following facts should be noted in regard to the curves in general that were obtained by the three analyses, with reference to Figure 3:

(i) For the case $S > 0$, the coefficient of S in (2.46) is always positive for $\frac{1}{1+k} \le i_x \le \frac{1}{1-k}$ and $0 \le k \le 1$. Thus when $S > 0$, Equation (2.44) becomes

$$S > \frac{(1 + k)i_x}{1 + 2ki_x^2 - i_x - ki_x} \ , \qquad (2.47)$$

[1]The term AB n^2 should be added to the right side of Routh's inequality, in his notation.

which is a necessary condition for infinitesimal stability. In addition, the function represented by the right hand side of (2.45) always has a maximum at $i_x = \frac{1}{\sqrt{2k}}$.

(ii) The two curves defined by condition (2.45) are identical to the curves given by Rumiantsev's Lyapunov analysis. But now, we gain a new necessary condition for infinitesimal stability, viz., that we are merely outside the two bounding curves $S = \frac{1}{1-i_x}$ and $S = \frac{1}{1-ki_x}$. This condition includes regions (c) and (f).

(iii) All three curves of Equation (2.44-2.46) intersect at the point

$$(\bar{i}_x, \bar{S}) = \left(\frac{1 + k + \sqrt{1-k}}{k(k-3)}, \; \frac{1}{1 - k\,\bar{i}_x}\right),$$

which makes the instability zones very clearly defined. The same curves also intersect at another point always lying outside and to the left of the range of possible rigid bodies.

(iv) The intersection point common to all of Routh's curves moves outside the region of possible rigid bodies $\left(\text{i.e., } \bar{i}_x \text{ becomes } = \frac{1}{1-k}\right)$ at the value of k given by the positive root of $4k^3 + 13k^2 + 2k - 3 = 0$, which is $\bar{k} = 0.39039$ to five digits. Thus, region (c) exists only for $k > \bar{k}$.

(v) For $S < 0$, region (f) disappears if $1/k \geq \frac{1}{1-k}$ (i.e. $k \leq 0.5$).

(vi) For the special case where the mass center G and the fixed point O coincide (the torque free Euler case) interesting results are obtained. Letting $d \to 0$ (i.e. $S \to \pm\infty$), we note in Figure 3 that the curves indicate stable spin about the maximum ($i_x < 1$) and minimum

($i_x > 1/k$) inertia axes and unstable spin about the intermediate ($1 < i_x < 1/k$) axis. This limiting case result is of course valid for the full-nonlinear system since it may be obtained by a Lyapunov analysis, see, e.g. [7] pages 238 and 239. It raises strong suspicions that the zones (c) and (f) may in fact be stable in the large which could be of considerable practical interest.

We conclude here that the present Floquet solution together with Rumiantsev's Lyapunov analysis and the corrected version of Routh's solution has clearly defined all possibilities with regard to stability and instability zones of the S versus i_x parameter planes for various k-values.

CHAPTER III

STABILITY OF A GYROSTAT WITH A FIXED POINT

3.1 Derivation of Equations of Motion

Consider a system of a finite number of rigid bodies $B_1,\ldots,B_n$ coupled together in order to form a gyrostat $\mathcal{G}$. Let B_1 have a point O fixed in an inertial frame $\mathcal{F}$ constituted by axes (X,Y,Z) as shown in Figure 4. Orthogonal axes (x_o, y_o, z_o) are permanently fixed in body B_1 and are principal for $\mathcal{G}$. Let I_{x_o}, I_{y_o}, and I_{z_o} be the corresponding principal moments of inertia of $\mathcal{G}$ with respect to the fixed point O. The mass center G of the gyrostat is assumed to lie on the z_o-axis. Bodies $B_1,\ldots,B_2$ are dynamically equivalent to axisymmetric bodies whose axes of symmetry are assumed to be parallel with the z_o-axis of B_1, to contain their mass centers, and to be permanently fixed in B_1.

If $(\hat{i}, \hat{j}, \hat{k})$ denote unit vectors along the respective axes (x_o, y_o, z_o) which are fixed in B_1, the angular velocity of B_1 in the inertial frame $\mathcal{F}$ can be represented by

$$
{}^{\mathcal{F}}\vec{\omega}^{B_1} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \quad . \tag{3.1}
$$

From the theorem in Appendix A, it follows that we can represent the angular momentum of $\mathcal{G}$ relative to O by $\vec{H}_o^* + \vec{h}$, where $\vec{H}_o^*$ is the angular momentum with respect to O of the entire system $\mathcal{G}$ considered as one solid body and $\vec{h}$ is the sum of the angular momenta of $B_2,\ldots,B_n$ in their motion relative to B_1, each taken with respect to its own mass

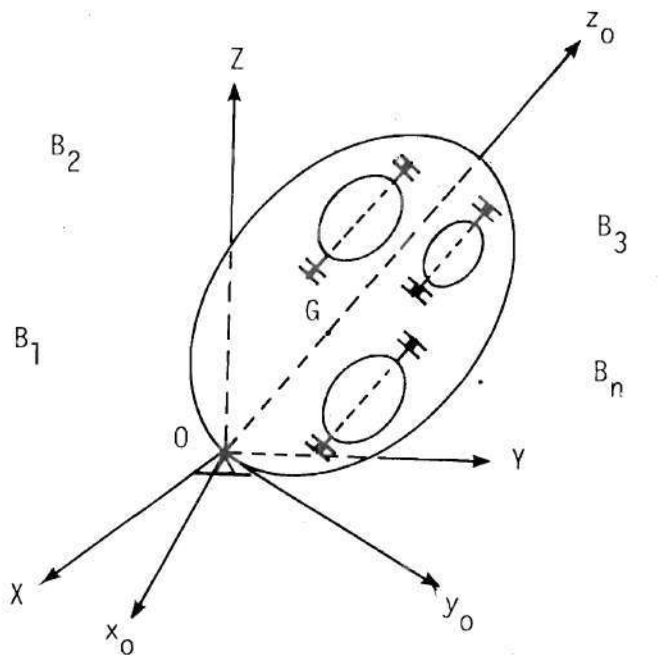

Figure 4. Gyrostat Consisting of Several Rigid Bodies.

center.

Recalling that I_{x_o}, I_{y_o} and I_{z_o} are the principal moments of inertia of G for point O, we observe that

$$\vec{H}_o^{*} = I_{x_o}\, \omega_x\, \hat{i} + I_{y_o}\, \omega_y\, \hat{j} + I_{z_o}\, \omega_z\, \hat{k}\ . \tag{3.2}$$

If $C_2,\ldots,C_n$ denote the moments of inertia of $B_2,\ldots,B_n$ about their respective axes fixed in B_1 and $\omega_2,\ldots,\omega_n$ are the corresponding relative spins of $B_2,\ldots,B_n$ with respect to B_1, it follows immediately that

$$\vec{h} = \Big(\sum_{i=2}^{n} c_i \omega_i\Big)\, \hat{k}\ . \tag{3.3}$$

In this analysis we are considering only the case where ω_i = const. ($i = 2,\ldots,n$), which was the assumption made by Rumiantsev

[17], Anchev [21,22], Kolesnikov [23], and others in similar investigations of the stability of gyrostats. The effects of the control system used to maintain the constant ω_i's was not considered in the analyses conducted by these investigators and is not considered in the present investigation. Kane [25] has shown that for the case of the torque-free gyrostat that the same stability results are obtained for a free rotor as far as a driven one. Hence, assuming constant ω_i's, we introduce equivalent inertia and angular velocity terms c^* and ω defined by

$$C^* = \sum_{i=2}^{n} C_i \quad , \tag{3.4}$$

$$\omega = \frac{\sum_{i=2}^{n} C_i \omega_i}{\sum_{i=2}^{n} C_i} \quad . \tag{3.5}$$

Therefore Equation (3.3) simplifies to

$$\vec{h} = c^* \omega \hat{k} \; . \tag{3.6}$$

Thus without loss of generality, we can represent all of the (dynamically equivalent) axisymmetric bodies $B_2, \ldots, B_n$ as just one axisymmetric body B^* with mass center G^* on $\ell\ell'$ as shown in Figure 5. Line $\ell\ell'$ is, of course, parallel to the z_o-axis. Therefore, making use of the theorem of Appendix A, we write

$$\overset{G}{\vec{H}_o} = \vec{H}^*_o + \vec{h} \; , \tag{3.7}$$

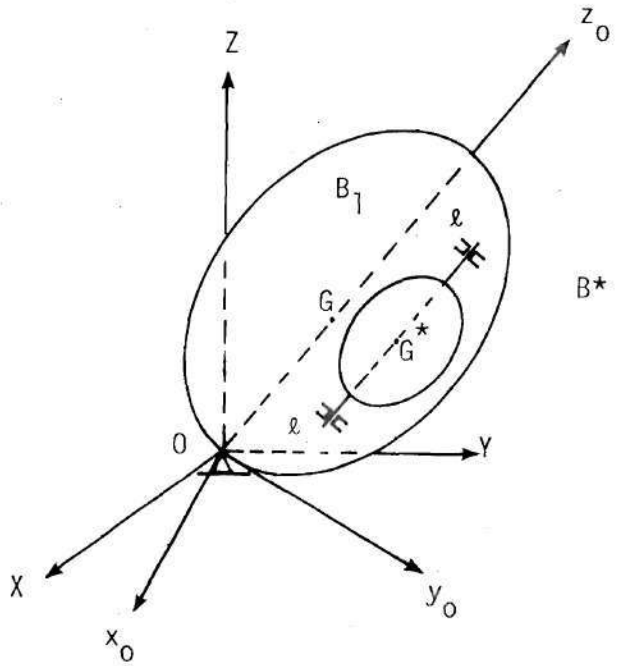

Figure 5. Gyrostat Consisting of Only Two Rigid Bodies.

where $\vec{H}_O^G$ is the total angular momentum about O of the gyrostat of the present problem.

Let us assume that the only external moment acting on G about the fixed point O are those due to uniform gravity. Thus if $\vec{M}_O$ denotes this resultant moment with respect to O, then by the principal of angular momentum

$$\overset{F}{\dot{\vec{H}}}{}_O^G = \vec{M}_O \ , \tag{3.8}$$

where $\overset{F}{(\dot{\ })}$ indicates differentiation with respect to time in the inertial frame F.

If $\hat{K}$ is a vertical unit vector, positive upward, then

$$\vec{M}_O = d\hat{k} \times (-\ mg\ \hat{K}) \quad , \tag{3.9}$$

where m is the total mass of the gyrostat G, g is the gravitational acceleration (assumed to be uniform), and d is the position of G (the mass center of G) on the z_o-axis. Thus writing $\hat{K}$ in terms of its direction cosines with respect to the axes (x_o, y_o, z_o) which are fixed in B_1 we obtain

$$\hat{K} = \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k} , \qquad (3.10)$$

where

$$\gamma_1 = \cos (x_o,Z)^1, \quad \gamma_2 = \cos (y_o,Z), \quad \text{and} \quad \gamma_3 = \cos (z_o,Z).$$

The left hand side of Equation (3.8) can be written as

$$ {}^{F}\dot{\vec{H}}{}^{G}_{o} = {}^{B_1}\vec{H}{}^{G}_{o} + {}^{F}\vec{\omega}^{B_1} \times \vec{H}{}^{G}_{o} \qquad (3.11)$$

where ${}^{B_1}(\dot{\ })$ indicates differentiation with respect to time in the moving frame B_1.

Now, substituting Equations (3.9) - (3.11) into Equation (3.8) gives rise to the Euler equations:

$$I_{x_o} \omega_x + (I_{z_o} - I_{y_o})\omega_y \omega_z + c^* \omega \omega_y = mgd\gamma_2 \qquad (3.12)$$

$$I_{y_o} \dot{\omega}_y + (I_{x_o} - I_{z_o})\omega_z \omega_x - c^* \omega \omega_x = - mgd\gamma_1 \qquad (3.13)$$

[1]Here cos (x_o,Z) means the cosine of the angle formed by the vertical Z axis and the x_o-axis which is fixed in body B_1.

$$I_{z_o} \dot{\omega}_z + (I_{y_o} - I_{x_o})\omega_x\omega_y = 0 \quad . \tag{3.14}$$

Also, if we differentiate $\hat{K}$ with respect to time in the inertial frame F, as was done in Chapter II, we again obtain Poisson's equations:

$$\dot{\gamma}_1 = \omega_z\gamma_2 - \omega_y\gamma_3 \quad , \tag{3.15}$$

$$\dot{\gamma}_2 = \omega_x\gamma_3 - \omega_z\gamma_1 \quad , \tag{3.16}$$

$$\dot{\gamma}_3 = \omega_y\gamma_1 - \omega_x\gamma_2 \quad . \tag{3.17}$$

Equations (3.12-3.17) constitute a set of six homogeneous non-linear differential equations in the variables ω_x, ω_y, ω_z, γ_1, γ_2, and γ_3 which determine the motion of the gyrostat.

3.2. Linearized Stability Analysis

An equilibrium solution of Equations (3.12-3.17) is seen to be

$$\omega_x = \omega_y = \gamma_1 = \gamma_2 = 0, \quad \omega_z = \Omega, \quad \gamma_3 = 1, \tag{3.18}$$

where Ω = const. is the unperturbed initial spin speed of body B_1 about the z_o-axis which is initially vertical.

Now, let us consider small perturbations from the equilibrium position of the form:

$$\begin{aligned} \omega_x &= \xi_1 \ , \quad \omega_y = \xi_2 \ , \quad \omega_n = \Omega + \xi_3 \ , \\ \gamma_1 &= \eta_1 \ , \quad \gamma_3 = \eta_2 \ , \quad \gamma_3 = 1 + \eta_3 \ , \end{aligned} \tag{3.19}$$

in which ξ_1, ξ_2, ξ_3, η_1, η_2 and η_3 are functions of time that must, of course, satisfy the Euler and Poisson equations throughout the ensuing motion following the disturbance.

Substituting these perturbations into Equations (3.12-3.17) and linearizing the resulting equations in the perturbations yields:

$$I_{x_o} \dot{\xi}_1 + [(I_{z_o} - I_{y_o})\Omega + c^* \omega]\xi_2 = mgd\ \eta_2 \quad , \tag{3.20}$$

$$I_{y_o} \dot{\xi}_2 + [(I_{x_o} - I_{z_o})\Omega - c^* \omega]\xi_1 = -\ mgd\ \eta_1 \ , \tag{3.21}$$

$$\dot{\eta}_1 = -\ \xi_2 + \Omega\ \eta_2 \quad , \tag{3.22}$$

$$\dot{\eta}_2 = \xi_1 - \Omega\ \eta_1 \ . \tag{3.23}$$

Two other differential equations are obtained which are solvable in closed form, having solutions ξ_3 = const. and η_3 = const., which are clearly bounded in time.

In this analysis, it is the stability of the linearized system of Equations (3.20-3.23) that we wish to examine. However, we must keep in mind that this is a linearized analysis, hence only the instability results can be extended to the full nonlinear system. We are not concerned with obtaining all the solutions of Equations (3.20-3.23); rather, we are seeking all conditions for which any existing solution will become unbounded. Toward this end we let

$$\begin{aligned} \xi_1 &= A_1 e^{\lambda t} \ , & \eta_1 &= A_3 e^{\lambda t} \ , \\ \xi_2 &= A_2 e^{\lambda t} \ , & \eta_2 &= A_4 e^{\lambda t} \ . \end{aligned} \tag{3.24}$$

Substituting Equation (3.24) into Equations (3.20-3.23) gives us four linear homogeneous algebraic equations:

$$I_{x_o} \lambda A_1 + [(I_{z_o} - I_{y_o})\Omega + c^* \omega]A_2 - mgd\, A_4 = 0 \quad , \tag{3.25}$$

$$[(I_{x_o} - I_{z_o})\Omega - c^* \omega]A_1 + I_{y_o} \lambda A_2 + mgd\, A_3 = 0 \quad , \tag{3.26}$$

$$A_2 + \lambda A_3 - \Omega A_4 = 0 \ , \tag{3.27}$$

$$- A_1 + \Omega A_3 + \lambda A_4 = 0 \ . \tag{3.28}$$

In order for this set of algebraic equations to have a nontrivial solution, the determinant of the coefficients of the A_i's must vanish. This condition leads us to the following characteristic equation:

$$\begin{aligned} I_{x_o} I_{y_o} \lambda^4 + \{ I_{x_o} I_{y_o} \Omega^2 - (I_{x_o} + I_{y_o}) mgd \\ + [(I_{z_o} - I_{y_o})\Omega + c^* \omega][(I_z - I_{x_o})\Omega + c^* \omega]\}\lambda^2 \\ + [(I_{z_o} - I_{y_o})\Omega^2 + c^* \omega\Omega - mdg][(I_{z_o} - I_{x_o})\Omega^2 \\ + c^* \omega\Omega - mgd] = 0 \quad , \end{aligned} \tag{3.29}$$

the roots of which determine the stability of the linearized system.

Equation (3.29) is a quartic equation in λ^2 of the form

$$a(\lambda^2)^2 + b\lambda^2 + c = 0 \ , \tag{3.30}$$

where

$$a = I_{x_o} I_{y_o} ,$$

$$b = I_{x_o} I_{y_o} \Omega^2 - (I_{x_o} + I_{y_o}) mgd$$

$$+ [(I_{z_o} - I_{y_o})\Omega + c^* \omega][(I_{z_o} - I_{x_o})\Omega + c^* \omega] ,$$

$$c = [(I_{z_o} - I_{y_o})\Omega^2 + c^* \omega\Omega - mgd][(I_{z_o} - I_{x_o})\Omega^2$$

$$+ c^* \omega\Omega - mgd] . \tag{3.31}$$

The roots (λ_i, i = 1,2,3,4) of the characteristic equation represented by Equation (3.30) are given by

$$\lambda_{1,2,3,4} = \pm \left\{ \frac{-b}{2a} \pm \left[\left(\frac{b}{2a} \right)^2 - \frac{c}{a} \right]^{1/2} \right\}^{1/2} . \tag{3.32}$$

For stability of the linearized system of Equations (3.25-3.28), the λ's defined by Equation (3.32) must be distinct and be purely imaginary; otherwise, instability will occur.

Therefore, for stability of the linearized system all of the following conditions must be satisfied:

(i) $b > 0$,

(ii) $c > 0$,

(iii) $b^2 - 4ac > 0$. (3.33)

Violation of any of the conditions of (3.33) will result in instability of the linear system (hence, the full nonlinear system) with the ξ_i and η_i of (3.24) having at least one term increasing without bound with time.

Hence, from (3.31) the stability conditions given by (3.33) clearly become

$$[(I_{z_o} - I_{y_o})\Omega^2 + c^*\omega\Omega - mdg][(I_{z_o} - I_{x_o})\Omega^2 + c^*\omega\Omega - mgd] > 0 \;, \qquad (3.34)$$

$$I_{x_o} I_{y_o}\Omega^2 - (I_{x_o} + I_{y_o})mgd$$

$$+ [(I_{z_o} - I_{y_o})\Omega + c^*\omega][(I_{z_o} - I_{x_o})\Omega + c^*\omega] > 0 \;, \qquad (3.35)$$

$$\{I_{x_o} I_{y_o}\Omega^2 - (I_{x_o} + I_{y_o})mgd$$

$$+ [(I_{z_o} - I_{y_o})\Omega + c^*\omega][(I_{z_o} - I_{x_o})\Omega + c^*\omega]\}^2$$

$$- 4I_{x_o} I_{y_o}[(I_{z_o} - I_{y_o})\Omega^2 + c^*\omega\Omega - mgd][(I_{z_o} - I_{x_o})\Omega^2$$

$$+ c^*\omega\Omega - mgd] > 0 \;. \qquad (3.36)$$

Following the notation of Chapter II, we introduce the nondimensional inertia parameters $i_x = I_{x_o}/I_{z_o}$ and $k = I_{y_o}/I_{x_o}$. Hence, conditions (3.34-3.36) become

$$\left[1 - ki_x + \frac{c^*\omega}{I_{z_o}\Omega} - \frac{mgd}{I_{z_o}\Omega^2}\right]\left[1 - i_x + \frac{c^*\omega}{I_{z_o}\Omega} - \frac{mgd}{I_{z_o}\Omega^2}\right] > 0 \;, \qquad (3.37)$$

$$k i_x^2 - (1 + k) i_x \frac{mgd}{I_{z_o} \Omega^2} + \left[1 - k i_x + \frac{c^* \omega}{I_{z_o} \Omega}\right]\left[1 - i_x + \frac{c^* \omega}{I_{z_o} \Omega}\right] > 0 \quad , \tag{3.38}$$

$$\left\{k i_x^2 - (1 + k) i_x \frac{mgd}{I_{z_o} \Omega^2} + \left[1 - k i_x + \frac{c^* \omega}{I_{z_o} \Omega}\right]\left[1 - i_x + \frac{c^* \omega}{I_{z_o} \Omega}\right]\right\}^2$$

$$- 4 k i_x^2 \left[1 - k i_x + \frac{c^* \omega}{I_{z_o} \Omega} - \frac{mgd}{I_{z_o} \Omega^2}\right]\left[1 - i_x + \frac{c^* \omega}{I_{z_o} \Omega} - \frac{mgd}{I_{z_o} \Omega^2}\right] > 0 \; . \tag{3.39}$$

Without loss of generality we take $\Omega > 0$ and define

$$S^* = \frac{I_{z_o} \Omega^2}{mg|d|} \quad , \tag{3.40}$$

and

$$\beta = \frac{c^* \omega}{\sqrt{I_{z_o} \, mg|d|}} \; . \tag{3.41}$$

Thus, conditions (3.37-3.39) clearly become

$$\left(1 - k i_x + \frac{\beta}{\sqrt{S^*}} - \operatorname{sgn}(d) \frac{1}{S^*}\right)\left(1 - i_x + \frac{\beta}{\sqrt{S^*}} - \operatorname{sgn}(d) \frac{1}{S^*}\right) > 0 \quad , \tag{3.42}$$

$$k i_x^2 - \operatorname{sgn}(d) \cdot \frac{(1+k) i_x}{S^*} + \left(1 - k i_x + \frac{\beta}{\sqrt{S^*}}\right)\left(1 - i_x + \frac{\beta}{\sqrt{S^*}}\right) > 0 \; , \tag{3.43}$$

$$\left\{ki_x^2 - \text{sgn}(d) \cdot \frac{(1+k)i_x}{S^*} + \left(1 - ki_x + \frac{\beta}{\sqrt{S^*}}\right)\left(1 - i_x + \frac{\beta}{\sqrt{S^*}}\right)\right\}^2$$

$$- 4\, ki_x^2\left(1 - ki_x + \frac{\beta}{\sqrt{S^*}} - \text{sgn}(d) \cdot \frac{1}{S^*}\right)\left(1 - i_x + \frac{\beta}{\sqrt{S^*}}\right.$$

$$\left. - \text{sgn}(d) \cdot \frac{1}{S^*}\right) > 0 \quad , \tag{3.44}$$

where

$$\text{sgn}(d) = \frac{d}{|d|} \quad . \tag{3.45}$$

Multiplying Equations (3.42 and 3.44) by S^{*2} and Equation (3.43) by S^* which is positive by definition, we obtain

$$[(1 - ki_x)S^* + \beta\sqrt{S^*} - \text{sgn}(d)][(1 - i_x)S^* + \beta\sqrt{S^*} - \text{sgn}(d)] > 0 \tag{3.46}$$

$$[2ki_x^2 - (1 + k)i_x - 1]S^* + \beta[2 - (1 + k)i_x]\sqrt{S^*}$$

$$+ \beta^2 - \text{sgn}(d) \cdot (1 + k)i_x > 0 \tag{3.47}$$

$$\{[2ki_x^2 - (1 + k)i_x + 1]S^* + \beta[2 - (1 + k)i_x]\sqrt{S^*} + \beta^2$$

$$- \text{sgn}(d) \cdot (1 + k)i_x\}^2 - 4ki_x^2[(1 - i_x)S^* + \beta\sqrt{S^*}$$

$$- \text{sgn}(d)][(1 - ki_x)S^* + \beta\sqrt{S^*} - \text{sgn}(d)] > 0 \quad . \tag{3.48}$$

Adopting the nondimensional spin speed parameter $S = \dfrac{I_{z_o}\Omega^2}{mgd}$ which was used in Chapter II, we clearly observe that

$$S = \mathrm{sgn}(d) \cdot S^* \quad . \tag{3.49}$$

Thus, any point (k, s, i_x) failing to satisfy all of the conditions (3.46-3.48) is an unstable point for the linearized system, and hence for the full nonlinear system. Also, we note that satisfaction of all of these conditions implies at least infinitesimal stability (i.e., stability of the linearized system).

Note that if we allow the relative spin ω of B^* to vanish (i.e., set $\beta = 0$) and use Equation (3.49), conditions (3.46-3.48) reduce to

$$[(1 - i_x)S - 1][1 - ki_x)S - 1] > 0 \ , \tag{3.50}$$

$$\{[2ki_x^2 - (1 + k)i_x + 1]S - (1 + k)i_x\} \cdot \mathrm{sgn}(d) > 0 \ , \tag{3.51}$$

$$\{[2ki_x^2 - (1 + k)i_x + 1]S - (1 + k)i_x\}^2 - 4ki_x^2[(1 - i_x)S - 1][(1 - ki_x)S - 1] > 0 \quad . \tag{3.52}$$

These stability conditions correspond, as should be expected, with the stability conditions (2.43-2.45) obtained in Chapter II from the linearized equations for an unsymmetrical rigid body spinning about an axis through a fixed point.

Now let us consider the case $\omega \neq 0$ (i.e., $\beta \neq 0$) which results in a gyrostat whose stability for the linearized equations of motion is determined by conditions (3.46-3.48). Here, we wish to investigate the effects of the relative motion on the respective stability-instability regions defined by conditions (3.50-3.52) corresponding to $\beta = 0$.

In the present analysis we must note that $\beta > 0$ implies that the relative spin ω of B^* with respect to B has the same sense as the spin Ω of B_1 in the inertial frame F, while $\beta < 0$ implies that ω has the opposite sense of Ω. To examine the effects of the relative spin on the stability of the system, we again choose the inertia imbalance ratio of $k = 0.7$ which was used in Chapter II for the unsymmetrical rigid body. Representative values of β are taken to be -1.0, 0.0, 1.0 and 2.0. The stability-instability regions for these values of β are given in Figures 6-9.

3.3 Rumiantsev's Lyapunov Analysis

Using the second Method of Lyapunov, Rumiantsev [17] was able to obtain a sufficient condition for the stability of the equilibrium solution (3.18) of the full nonlinear equations (Equations (3.12-3.17)) of the present problem. Converting his results to the notation used here we have

$$(1 - i_x)S^* + \beta\sqrt{S^*} - \mathrm{sgn}(d) > 0 \quad , \tag{3.53}$$

which defines portions of the stability regions obtained by the linear analysis of the previous section (3.2). In Figures 6-9, these regions are labeled "LYAP", indicating stability in the sense of Lyapunov (stability for the full nonlinear system).

3.4 General Comments

In the present analysis, the results show that the inclusion of a symmetric spinning body or bodies inside a single rigid body with a fixed point can have both beneficial and harmful effects on the stability of the system.

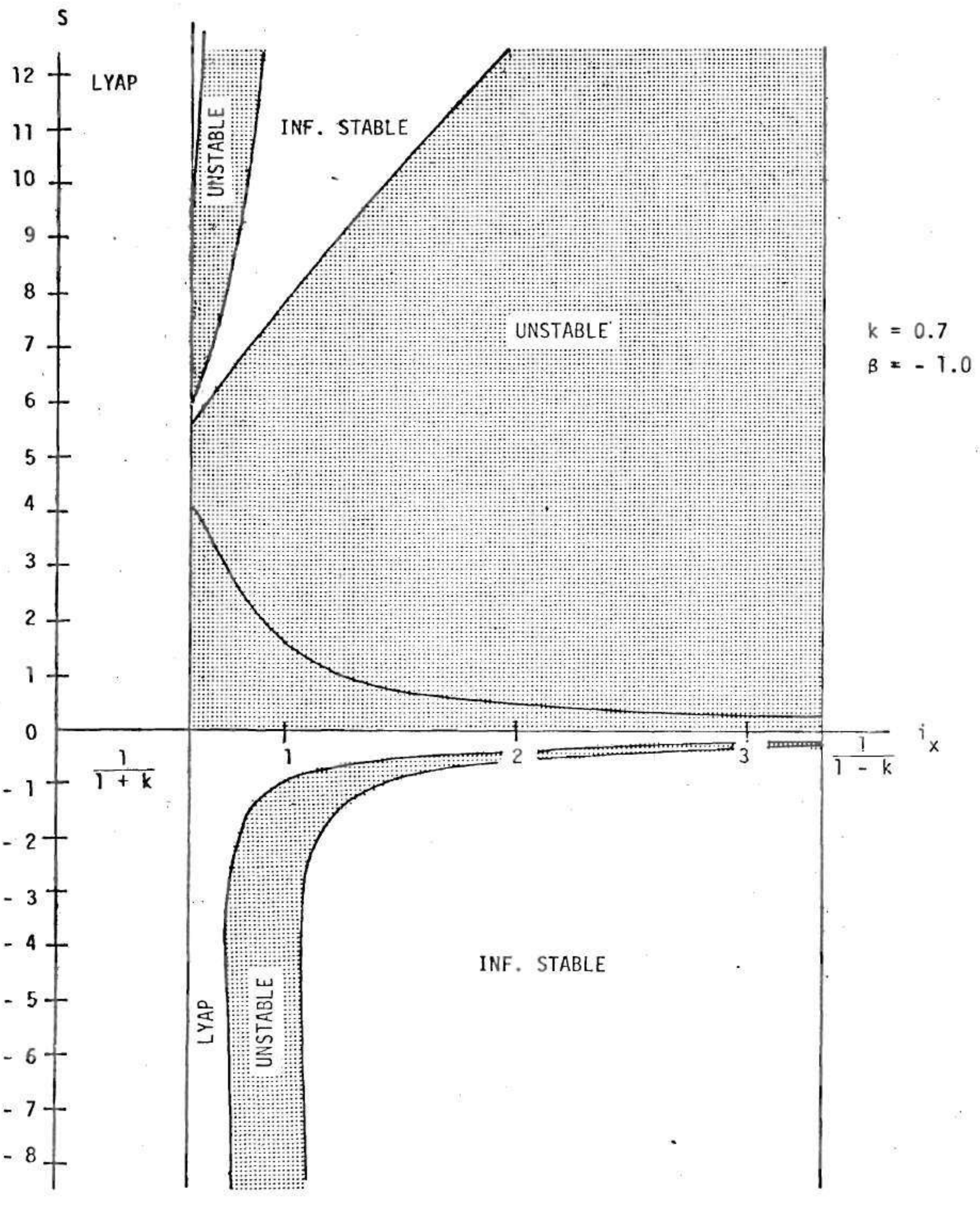

Figure 6. Stability Regions for a Gyrostat,
k = 0.7, = - 1.0.

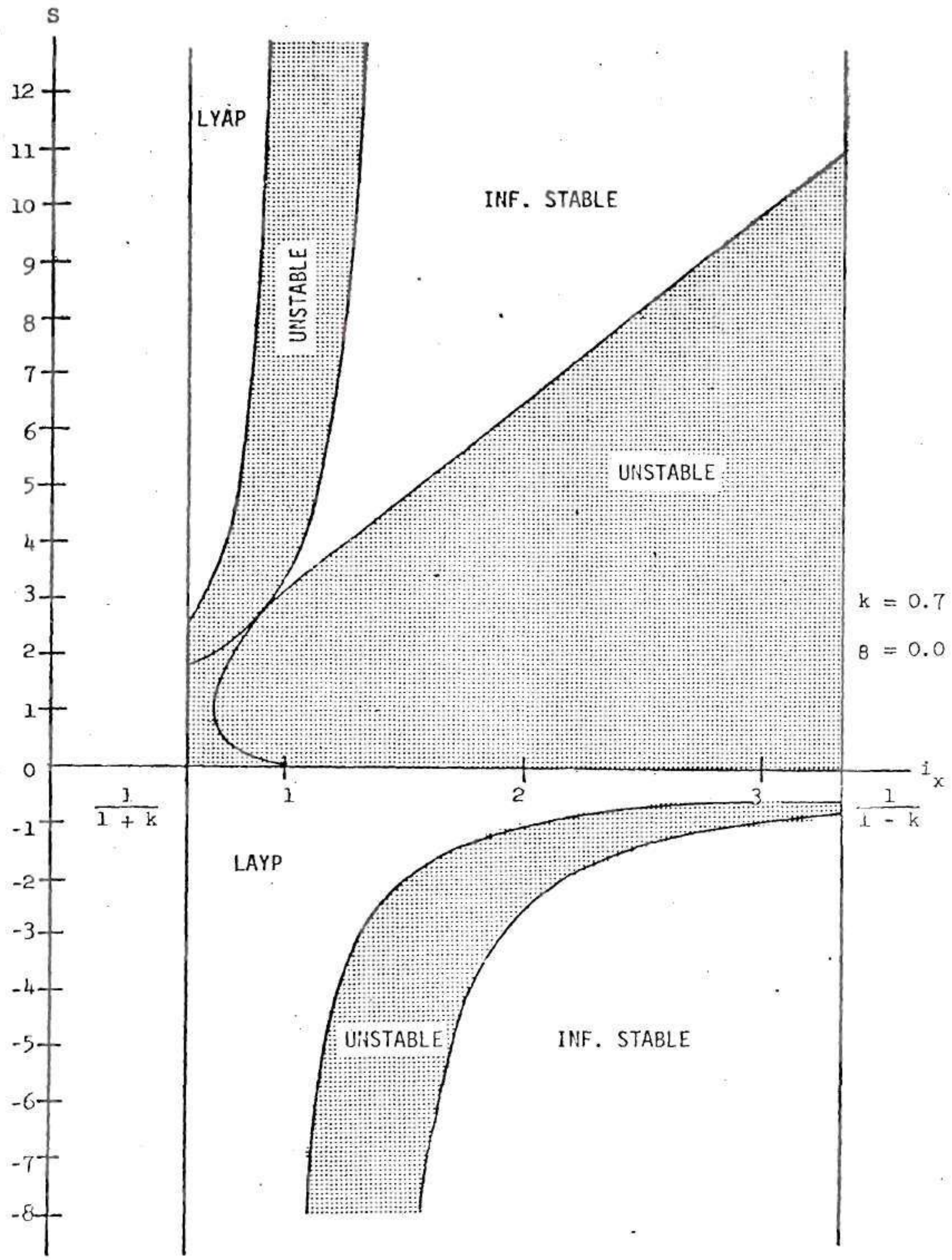

Figure 7. Stability-instability Regions for a Gyrostat, k = 0.7, = 0.0.

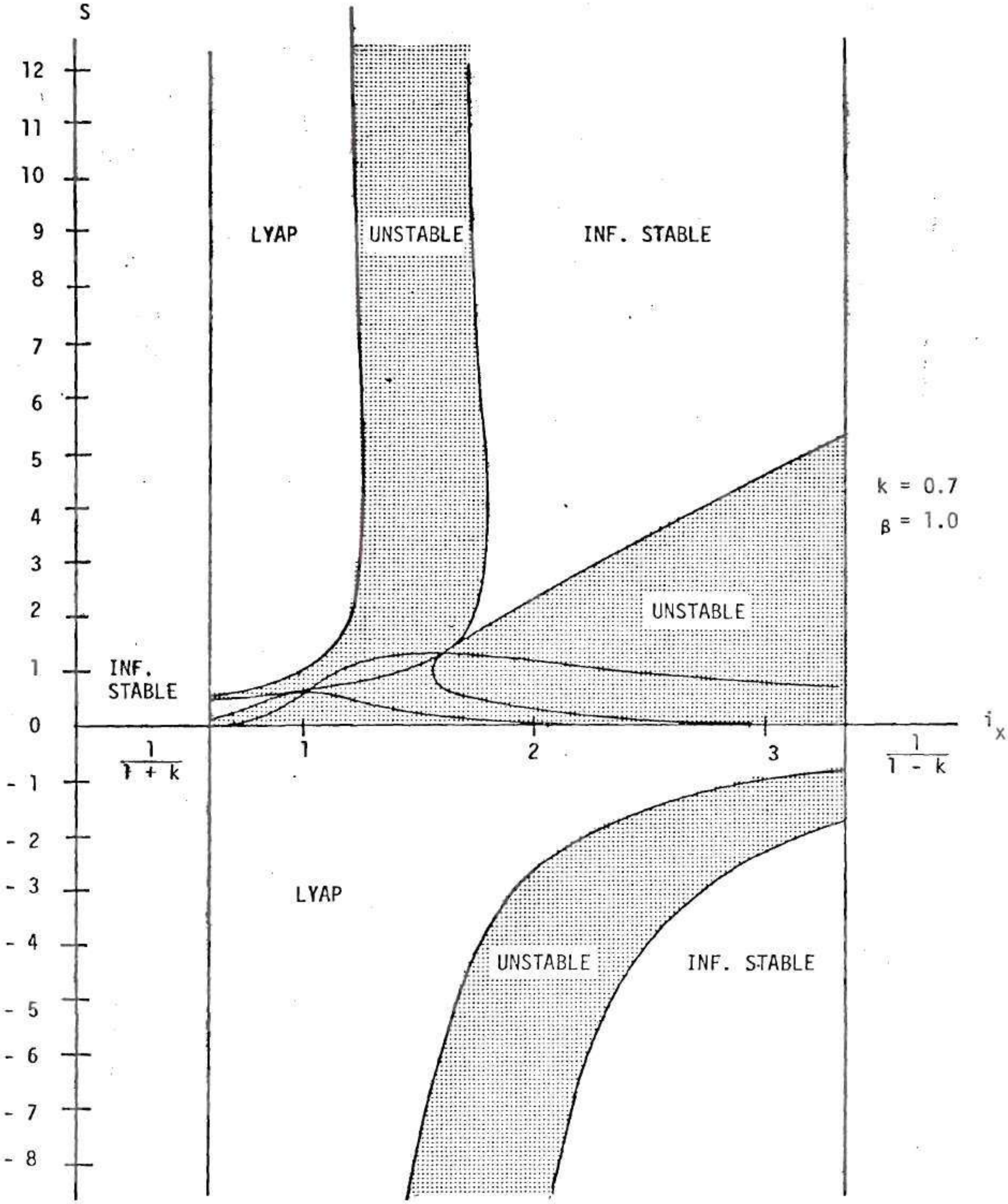

Figure 8. Stability-instability Regions for a Gyrostat,
k = 0.7, = 1.0.

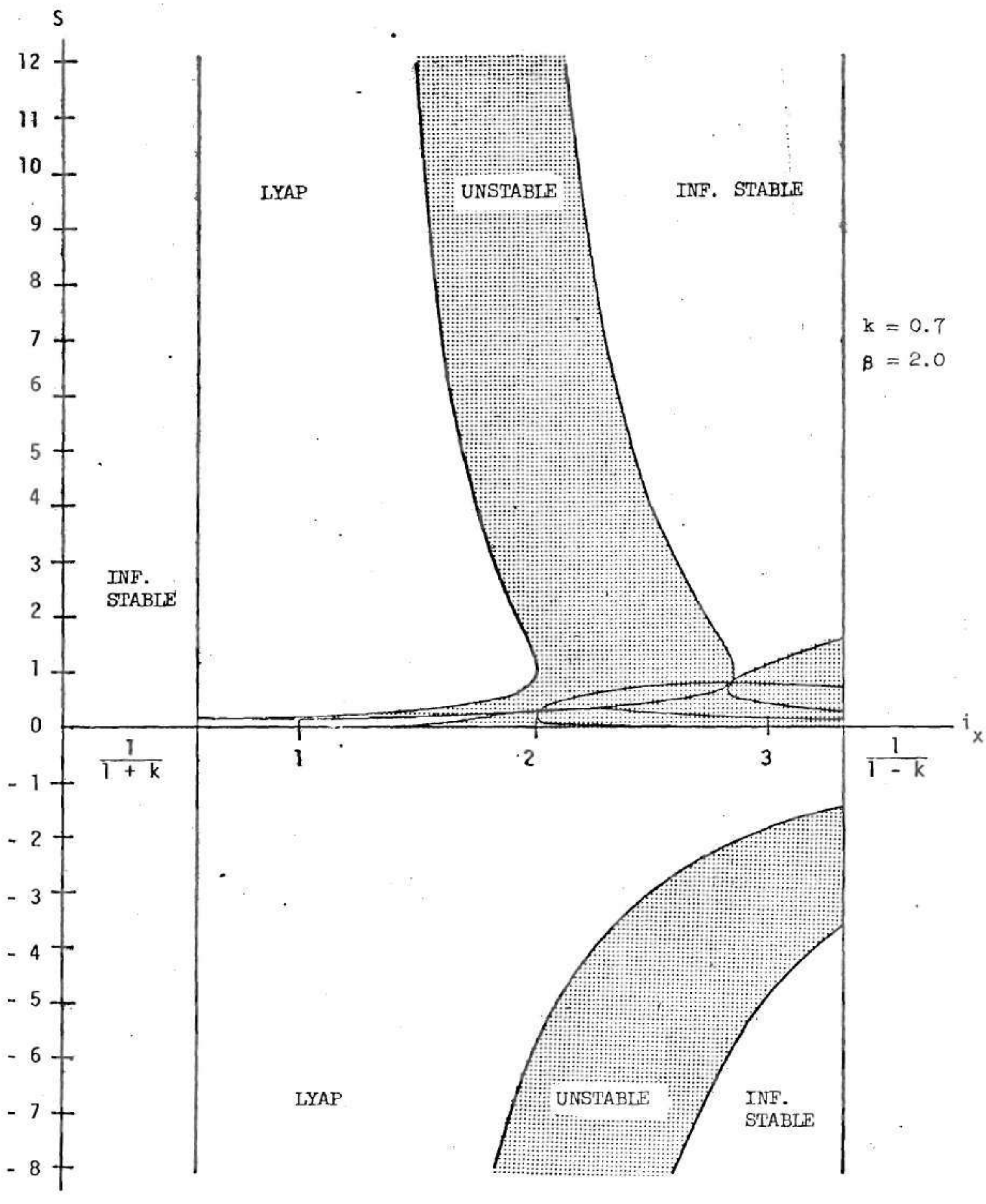

Figure 9. Stability-instability Regions Gyrostat,
$k = 0.7$, $\beta = 2.0$.

In Figures 6 through 9, those regions that were shown to be unstable for the linear system, and hence also for the nonlinear system, are labeled "UNSTABLE": while those regions that were found to be stable for the linearized equations but could not be shown to be Lyapunov stable are labeled "INF. STABLE", indicating infinitesimal stability. Zones that were found to be stable for the full nonlinear system by the direct method of Lyapunov are labeled "LYAP", indicating stability in the sense of Lyapunov.

As depicting in Figure 6, the introduction of negative relative spin ($\beta < 0$) produces deleterious effects on the stability of the system. The curves of Figure 6 clearly show that for relatively small values of positive S much of the Lyapunov stable region has become unstable; also, the infinitesimally stable region has decreased in size and shifted to the left. However, for small values of negative S, the reduction in the size of the Lyapunov stable region is followed by an increase in the size of the infinitesimally stable region. Nevertheless, it is the effect on the Lyapunov stable regions which is of greater importance, since it is known to be stable for the full nonlinear system.

Figures 8 and 9 depict the effects of positive relative spin on the gyrostat of the present analysis. It is clear from both figures that for small values of S, both positive and negative, the addition of positive relative spin results in an increase in the size of the Lyapunov stable regions, also the unstable region lying just below the upper infinitesimally stable region decreases with increasing β. One should also note (in Figures 8 and 9) the introduction of a new infinitesimally stable region for positive S. This new zone lies just below the upper Lyapunov stable region and near the left hand rigid body limit, $i_x = \frac{1}{1+k}$. Although this new infinitesimally stable region is quite small, it seems to grow in width with β.

CHAPTER IV

AN INSTABILITY ANALYSIS OF AN EXTENDED GYROSTAT VIA FLOQUET'S THEORY

4.1 Derivation of Equations of Motion

In Figure 10, we have an extended gyrostat G^* consisting of two coupled rigid bodies B_1 and B_2 with B_1 having a point O which is fixed in an inertial frame F constituted by axes (X,Y,Z). Orthogonal axes (x_o,y_o,z_o) are permanently fixed in body B_1, and they are principal

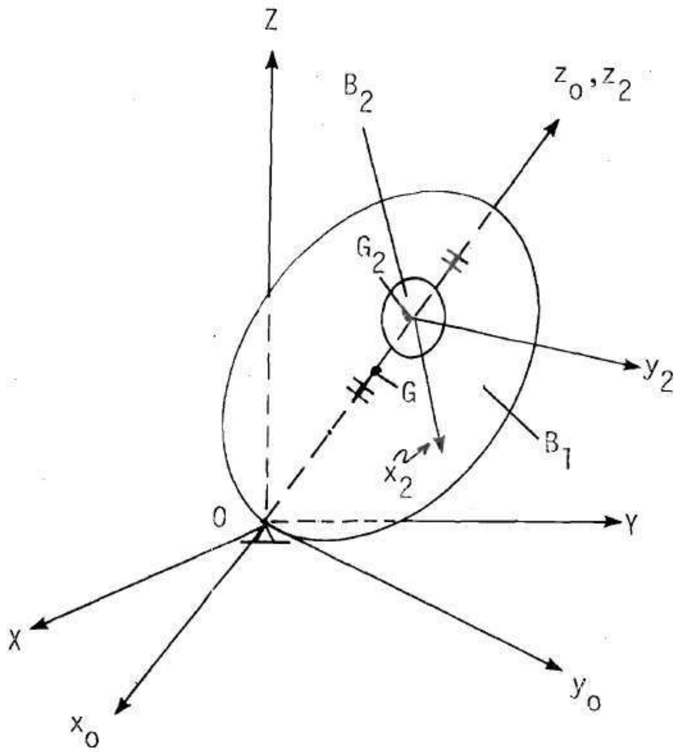

Figure 10. The Extended Gyrostat.

axes of B_1 for point O. The z_o-axis contains the mass center G_1 of body B_1 and also the mass center G_2 of body B_2. Permanently fixed in B_2 are orthogonal axes (x_2, y_2, z_2), having origin G_2 and being principal axes of B_2 for G_2. The z_o-axis of body B_1 and the z_2-axis of body B_2 are coincident and body B_2 is free to spin relative to B_1 about this common axis inside a cavity large enough so as not to prevent motion.

If we let $(\hat{i}, \hat{j}, \hat{k})$ be unit vectors parallel to the respective body fixed axes (x_o, y_o, z_o) of B_1, we can represent the angular velocity of B_1 in $\mathcal{F}$ by

$$ {}^{\mathcal{F}}\vec{\omega}^{B_1} = \omega_x \, \hat{i} + \omega_y \, \hat{j} + \omega_z \, \hat{k} \quad . \tag{4.1} $$

In this analysis, we are assuming that B_2 is spinning about the z_o-axis of B_1 at a constant angular speed of ω relative to B_1; hence, the angular velocity of B_2 in $\mathcal{F}$ obviously becomes

$$ {}^{\mathcal{F}}\vec{\omega}^{B_2} = \omega_x \, \hat{i} + \omega_y \, \hat{j} + (\omega_z + \omega)\hat{k} \quad . \tag{4.2} $$

Let $\vec{r}_p$ represent the vector from the fixed point O to an arbitrary point P of G^*. Then

$$ \vec{r}_p = \begin{cases} x \, \hat{i} + y \, \hat{j} + z \, \hat{k} & \text{for } P \in B_1 \\ d_{G_2} \, \hat{k} + \vec{\ell} & \text{for } P \in B_2 \quad , \end{cases} \tag{4.3} $$

where (x, y, z) are the coordinates of P when P is fixed in body B_1, d_{G_2}

is the location of the mass center G_2 of body B_2 on the z_o-axis, and $\vec{\rho}$ is a vector from G_2 to P when P is fixed in body B_2.

Thus if $(\hat{i}_2, \hat{j}_2, \hat{k}_2)$ denote the unit vectors parallel to the respective (x_2, y_2, z_2) axes of body B_2, we can express $\vec{\rho}$ as

$$\vec{\rho} = x_2\hat{i}_2 + y_2\hat{j}_2 + z_2\hat{k}_2 \quad , \tag{4.4}$$

in which (x_2, y_2, z_2) are the coordinates of P with respect to the axes fixed in B_2.

The velocity of the point P in the inertial frame F is given by

$$\vec{V}_p = {}^{F}\dot{\vec{r}}_p \quad , \tag{4.5}$$

where ${}^{F}\dot{\vec{r}}_p$ represents the time derivative of $\vec{r}_p$ in F. Therefore, depending upon the location of P in G^*, Equation (4.5) becomes

$$\vec{V}_p = \begin{cases} {}^{F}\vec{\omega}^{B_1} \times \vec{r}_p & \text{for } P \in B_1 \\ \\ {}^{F}\vec{\omega}^{B_2} \times \vec{r}_p & \text{for } P \in B_2 \quad , \end{cases} \tag{4.6}$$

where we note that O is a fixed point not only of body B_1 but also of body B_2 extended.

By definition the angular momentum of G^* with respect to O is given by

$$\vec{H}_o^{G^*} = \int_{G^*} (\vec{r}_p \times \vec{v}_p) dm_p \quad , \tag{4.7}$$

where dm_p is a differential element of mass containing P, and the integration is taken over the entire system G^*.

Since bodies B_1 and B_2 contain no common material point, we can write

$$\vec{H}_o^{G^*} = \vec{H}_o^{B_1} + \vec{H}_o^{B_2} , \tag{4.8}$$

in which

$$\vec{H}_o^{B_1} = \int_{B_1} (\vec{r}_p \times \vec{v}_p) dm_p , \tag{4.9}$$

and

$$\vec{H}_o^{B_2} = \int_{B_2} (\vec{r}_p \times \vec{v}_p) dm_p , \tag{4.10}$$

and where in Equations (4.9, 4.10) the integrations are taken over bodies B_1 and B_2 respectively.

Now substituting Equation (4.6) into Equation (4.9) yields

$$\vec{H}_o^{B_1} = \int_{B_1} \vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p) dm_p , \tag{4.11}$$

which with the use of an identity from vector calculus can be put in the form

$$\vec{H}_o^{B_1} = \int_{B_1} [(\vec{r}_p \cdot \vec{r}_p) \, {}^{F}\vec{\omega}^{B_1} - (\vec{r}_p \cdot {}^{F}\vec{\omega}^{B_1}) \vec{r}_p] dm_p . \tag{4.12}$$

Thus, employing Equations (4.1,4.3), we clearly obtain

$$
\begin{aligned}
\vec{H}_o^{B_1} &= \int_{B_1} [(x^2 + y^2 + z^2)(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}) \\
&\qquad - (x\omega_x + y\omega_y + z\omega_z)(x\hat{i} + y\hat{j} + z\hat{k})]dm_p \\
&= \int_{B_1} \{[y^2 + z^2)\omega_x - x_y\omega_y - x_z\omega_z]\hat{i} \\
&\qquad + [(z^2 + x^2)\omega_y - yz\omega_z - yx\omega_x]\hat{j} \\
&\qquad + [(x^2 + y^2)\omega_z - zx\omega_x - zy\omega_y]\hat{k}\}dm_p
\end{aligned}
\qquad (4.13)
$$

Since (x_o, y_o, z_o) were chosen to be the principal directions for B_1 with respect to O, then all of the products of inertia vanish. Hence by the definitions of moments and products of inertia, Equation (4.13) becomes

$$
\vec{H}_o^{B_1} = I_{x_o}\omega_x\hat{i} + I_{y_o}\omega_y\hat{j} + I_{z_o}\omega_z\hat{k} \ , \qquad (4.14)
$$

in which $(I_{x_o}, I_{y_o}, I_{z_o})$ are the principal moments of inertia of body B_1 for point O.

Also, substituting Equation (4.6) into Equation (4.10) gives

$$
\vec{H}_o^{B_2} = \int_{B_2} \vec{r}_p \times (\vec{\omega}^{F\,B_2} \times \vec{r}_p)dm_p \ . \qquad (4.15)
$$

Since P is contained in body B_2 in the integral of Equation (4.15), we see by using Equation (4.3) that

$$\vec{H}_o^{B_2} = \int_{B_2} (d_{G_2}\hat{k} + \vec{\rho}) \times [\,{}^{F}\vec{\omega}^{B_2} \times (d_{G_2}\hat{k} \times \vec{\rho})]dm_p$$

$$= \int_{B_2} [d_{G_2}\hat{k} \times ({}^{F}\vec{\omega}^{B_2} \times d_{G_2}\hat{k}) + d_{G_2}\hat{k} \times ({}^{F}\vec{\omega}^{B_2} \times \vec{\rho})$$

$$+ \vec{\rho} \times ({}^{F}\vec{\omega}^{B_2} \times d_{G_2}\hat{k}) + \vec{\rho} \times ({}^{F}\vec{\omega}^{B_2} \times \vec{\rho})]dm_p \,. \qquad (4.16)$$

The quantities ${}^{F}\vec{\omega}^{B_2}$ and $d_{G_2}\hat{k}$ are independent of the position of P, and thus can be treated as vector constants in the integration over the body B_2.

Therefore, Equation (4.16) can be rewritten as

$$\vec{H}_o^{B_2} = d_{G_2}\hat{k} \times ({}^{F}\vec{\omega}^{B_2} \times d_{G_2}\hat{k})m_2$$

$$+ d_{G_2}\hat{k} \times ({}^{F}\vec{\omega}^{B_2} \times \int_{B_2} \vec{\rho}\, dm_p)$$

$$+ (\int_{B_2} \vec{\rho}\, dm_p) \times ({}^{F}\vec{\omega}^{B_2} \times d_{G_2}\hat{k})$$

$$+ \int_{B_2} \vec{\rho} \times ({}^{F}\vec{\omega}^{B_2} \times \vec{\rho})dm_p \,, \qquad (4.17)$$

where

$$m_2 = \int_{B_2} dm_p \ .$$

Since G_2 is the mass center of body B_2 we have

$$\int_{B_2} \vec{\rho}\ dm = 0 \ . \tag{4.18}$$

Thus, Equation (4.17) can be reduced to

$$\vec{H}_o^{B_2} = d_{G_2}\hat{k} \times ({}^{F}\vec{\omega}^{B_2} \times d_{G_2}\hat{k})m_2 + \int_{B_2} \vec{\rho} \times ({}^{F}\vec{\omega}^{B_2} \times \vec{\rho})dm_p \ . \tag{4.19}$$

Figure 11 is a view down the positive z_o-axis which gives one a

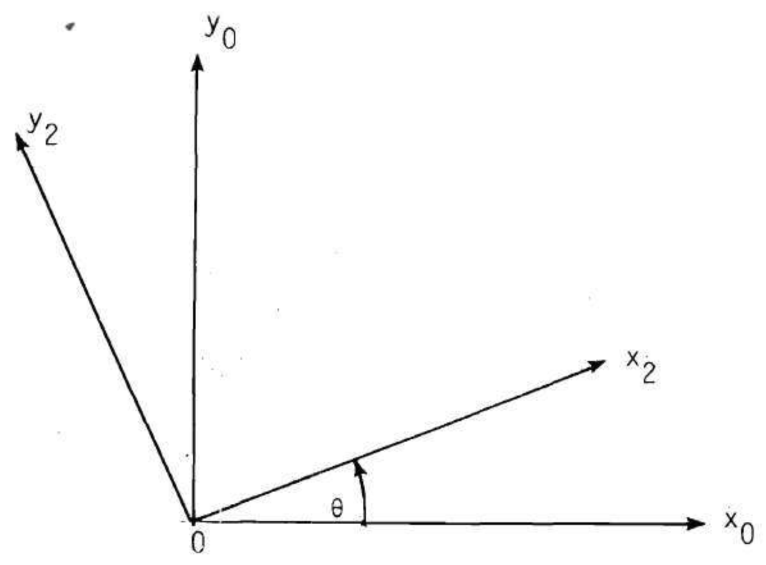

Figure 11. View Down the z_o-axis.

clear picture of the following relationships:

$$\hat{i}_2 = \cos\theta\hat{i} + \sin\theta\,\hat{j}\ ,$$
$$\hat{j}_2 = -\sin\theta\,\hat{i} + \cos\hat{\theta}\,j\ , \tag{4.20}$$

and clearly $\omega = \dot{\theta}$ for the simple rotational motion of B_2 with respect to B_1.

Hence, Equation (4.4) can be rewritten as

$$\vec{\rho} = (x_2c_\theta - y_2s_\theta)\hat{i} + (x_2s_\theta + y_2c_\theta)\hat{j} + z_2\hat{k}\ , \tag{4.21}$$

where as before $C_\theta = \cos\theta$ and $S_\theta = \sin\theta$.

Therefore, substituting Equation (4.2, 4.21) into Equation (4.19) and performing the indicated operations, we obtain

$$\begin{aligned}\vec{H}_o^{B_2} &= [(I_{x_2}C_\theta^2 + I_{y_2}S_\theta^2 + m_2d_{G_2}^2)\omega_x + (I_{x_2} - I_{y_2})S_\theta C_\theta\,\omega_y]\hat{i}\\ &+ [(I_{x_2} - I_{y_2})S_\theta C_\theta\omega_x + (I_{x_2}S_\theta^2 + I_{y_2}C_\theta^2 + m_2d_{G_2}^2)\omega_y]\hat{j}\\ &+ I_{z_o}(\omega_2 - \omega)\hat{k}\ ,\end{aligned} \tag{4.22}$$

where $(I_{x_2}, I_{y_2}, I_{z_2})$ are the principal moments of inertia of body B_2 for its mass center G_2 corresponding to the principal directions (x_2, y_2, z_2).

Hence, using Equations (4.13, 4.22) in Equation (4.2) we obtain

$$\vec{H}_o^{G^*} = [(I_{x_o} + I_{x_o}C_\theta^2 + I_{y_2}S_\theta^2 + m_2d_{G_2})\omega_x + (I_{x_2} - I_{y_2})S_\theta C_\theta\omega_y]\hat{i}$$

$$+ [(I_{x_2} - I_{y_2})S_\theta C_\theta \omega_x + (I_{y_o} + I_{x_2} S_\theta^2 + I_{y_2} C_\theta^2 + m_2 d_{G_2}^2)\omega_y]\hat{j}$$

$$+ [(I_{z_o} + I_{z_2})\omega_z + I_{z_2}\omega]\hat{k} , \qquad (4.23)$$

which represents the angular momentum of the entire extended gyrostat for the point O which is fixed in the inertial frame F.

Let us make the following simplifications:

$$I_x = I_{x_o} + I_{x_2} + m_2 d_{G_2}^2 , \qquad (4.24a)$$

$$I_y = I_{y_o} + I_{y_2} + m_2 d_{G_2}^2 , \qquad (4.24b)$$

$$I_z = I_{z_o} + I_{z_2} , \qquad (4.24c)$$

$$e = I_{x_2} - I_{y_2} , \qquad (4.24d)$$

where (I_x, I_y, I_z) correspond to the principal moments of inertia of G^* for the point O at a time when $\theta = 0$.

With the simplifications introduced by Equations (4.24a-4.24d), we can rewrite Equation (4.23) as

$$\vec{H}_o^{G^*} = [(I_x - e\, S_\theta^2)\omega_x + e\, S_\theta C_\theta \omega_y]\,\hat{i}$$

$$+ [e\, S_\theta C_\theta \omega_x + (I_y + e\, S_\theta^2)\omega_y]\,\hat{j}$$

$$+ [I_z \omega_z + I_{z_2}\omega]\,\hat{k} . \qquad (4.25)$$

In the present analysis, we are assuming that the only external moments acting on G^* about 0 are those due to uniform gravity. Since d denotes the position on the z-axis of the mass center G of the gyrostat G^*, then the moment $\vec{M}_o$ exerted on G^* about 0 is given by

$$\vec{M}_o = d\hat{k} \times (-mg\hat{k}) \quad , \tag{4.26}$$

where m is the total mass of G^*, g is the acceleration of gravity, and $\hat{K}$ is an upward unit vector. Writing $\hat{K}$ in terms of its direction cosines with respect to the axes (x_o, y_o, z_o) fixed in B_1, we have

$$\hat{K} = \gamma_1 \hat{i} + \gamma_2 \hat{j} + \gamma_3 \hat{k} \quad , \tag{4.27}$$

in which

$$\gamma_1 = \hat{i} \cdot \hat{K} \, , \qquad \gamma_2 = \hat{j} \cdot \hat{K} \, , \qquad \text{and} \qquad \gamma_3 = \hat{k} \cdot \hat{K}$$

The moment equation with respect to the fixed point 0 is

$$\overset{F}{\dot{\vec{H}}}{}_o^{G*} = \overset{B_1}{\dot{\vec{H}}}{}_o^{G*} + \overset{F}{\vec{\omega}}{}^{B_1} \times \vec{H}_o^{G*} = \vec{M}_o \quad . \tag{4.28}$$

Thus, if we substitute Equations (4.1,4.25,4.26) into Equation (4.28) we obtain the Euler equations for the present problem:

$$(I_x - eS_\theta^2)\dot{\omega}_x + \frac{1}{2} eS_{2\theta}\dot{\omega}_y - \frac{1}{2} eS_{2\theta}\omega_x\omega_z + (I_z - I_y - eS_\theta^2)\omega_y\omega_z$$

$$- eS_{2\theta}\omega\omega_x + (eC_{2\theta} + I_{z_2})\omega\omega_y = mgd \ \gamma_2 \quad , \tag{4.29}$$

$$\frac{1}{2}\, eS_{2\theta}\dot{\omega}_x + (I_y + eS_\theta^2)\dot{\omega}_y + \frac{1}{2}\, eS_{2\theta}\omega_y\omega_z + (I_x - I_z - eS_\theta^2)\omega_x\omega_z$$

$$+ (eC_{2\theta} - I_{z_2})\omega\omega_x + eS_{2\theta}\omega\omega_y = - \text{ mgd } \gamma_1 \ , \tag{4.30}$$

$$I_z\ \dot{\omega}_z + \frac{1}{2}\, eS_{2\theta}\omega_x^2 + [I_y - I_x + 2eS_\theta^2]\omega_x\omega_y - \frac{1}{2}\, eS_{2\theta}\omega_y^2 = 0 \ , \tag{4.31}$$

where

$$S_\theta = \sin\theta \ , \quad S_{2\theta} = \sin 2\theta \ , \quad \text{and} \quad C_{2\theta} = \cos 2\theta \ .$$

As before, the Poisson equations follow from the differentiation of the unit vector $\hat{K}$ with respect to time in the inertial frame F. These equations are

$$\dot{\gamma}_1 = \omega_z\gamma_2 - \omega_y\gamma_3 \ , \tag{4.32}$$

$$\dot{\gamma}_2 = \omega_x\gamma_3 - \omega_z\gamma_1 \ , \tag{4.33}$$

$$\dot{\gamma}_3 = \omega_y\gamma_1 - \omega_x\gamma_2 \ . \tag{4.34}$$

Equations (4.29-4.34) constitute the equations of motion for the present problem. Note that if we set $e = 0$ in Equations (4.29-4.34), we obtain

$$I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z + I_{z_2}\omega\omega_y = \text{mgd } \gamma_2 \ , \tag{4.35}$$

$$I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x - I_{z_2}\omega\omega_x = - \text{ mgd } \gamma_1 \ , \tag{4.36}$$

$$I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y = 0 \ , \tag{4.37}$$

which are the Euler equations for the type gyrostat of Chapter III.

4.2 Stability Analysis via Floquet's Theory

An equilibrium solution of Equations (4.29-4.34) is found to be

$$\omega_x = \omega_y = \gamma_1 = \gamma_2 = 0 \ , \quad \omega_z = \Omega, \quad \text{and} \quad \gamma_3 = 1 \ , \tag{4.38}$$

where Ω = const. is the unperturbed initial spin speed of B_1 about the vertical.

Now let us consider again small perturbations from this equilibrium position of the form:

$$\omega_x = \xi_1 \ , \quad \omega_y = \xi_2 \ , \quad \omega_z = \Omega + \xi_3 \ ,$$

and (4.39)

$$\gamma_1 = \eta_1 \ , \quad \gamma_2 = \eta_2 \ , \quad \gamma_3 = 1 + \eta_3 \ ,$$

where ξ_1, ξ_2, ξ_3, η_1, η_2 and η_3 are functions of time and must satisfy the Euler and Poisson equations (Equations (4.29-4.34)) of the present problem.

Substituting these perturbations into Equations (4.29-4.34) and linearizing the results in terms of the perturbations gives rise to the following equations, which are independent of ξ_3 and η_2:

$$(I_x - e\,S_\theta^2)\dot{\xi}_1 + \tfrac{1}{2}\,eS_{2\theta}\dot{\xi}_2 - \tfrac{1}{2}\,eS_{2\theta}\Omega\xi_1 + (I_z - I_y - eS_\theta^2)\Omega\xi_2$$

$$- eS_{2\theta}\omega\xi_1 + (eC_{2\theta} + I_{z_2})\omega\xi_2 = \text{mgd}\ \eta_2 \ , \tag{4.40}$$

$$\tfrac{1}{2}\,eS_{2\theta}\dot{\xi}_1 + (I_y + eS_\theta^2)\dot{\xi}_2 + \tfrac{1}{2}\,eS_{2\theta}\Omega\xi_2 + (I_x - I_z - eS_\theta^2)\Omega\xi_1$$

$$+ (eC_{2\theta} - I_{z_2})\omega\xi_1 + eS_{2\theta}\omega\xi_2 = -\ \text{mgd}\ \eta_1 \ , \tag{4.41}$$

$$\dot{\eta}_1 = \Omega\,\eta_2 - \xi_2\ , \tag{4.42}$$

$$\dot{\eta}_2 = \xi_1 - \Omega\,\eta_1\ . \tag{4.43}$$

We also obtain ξ_3 = const. and γ_3 = const. from the linearization of Equations (4.31, 4.34).

Equations (4.40,4.41) can be rewritten as

$$\begin{aligned}(I_x - eS_\theta^2)\dot{\xi}_1 + \tfrac{1}{2}\,eS_{2\theta}\dot{\xi}_2 = \left(\omega + \tfrac{1}{2}\,\Omega\right)eS_{2\theta}\xi_1 \\ - [(I_2 - I_y - eS_\theta^2)\Omega + (eC_{2\theta} + I_{z_2})\omega]\xi_2 \\ + mgd\ \gamma_2\ ,\end{aligned} \tag{4.44}$$

$$\begin{aligned}\tfrac{1}{2}\,eS_{2\theta}\dot{\xi}_1 + (I_y + eS_\theta^2)\dot{\xi}_2 = -\ [(I_x - I_z - eS_\theta^2)\Omega + (eC_{2\theta} - I_{z_2})\omega]\xi_1 \\ - \left(\omega + \tfrac{1}{2}\,\Omega\right)eS_{2\theta}\xi_2 - mgd\ \gamma_1\ .\end{aligned} \tag{4.45}$$

Let Δ represent the determinant of the coefficients of $\dot{\xi}_1$ and $\dot{\xi}_2$ on the left hand side of Equations (4.44, 4.45). Thus, after some mathematical manipulation we obtain

$$\Delta = I_x I_y + (I_x - I_y - e)eS_\theta^2\ , \tag{4.46}$$

where it can easily be shown that Δ is always positive.

Therefore with the use of Cramer's rule, Equations (4.44, 4.45) can be put into the following form:

$$\dot{\xi}_1 = \frac{1}{\Delta}\left\{(I_y + eS_\theta^2)\left(\omega + \frac{1}{2}\Omega\right) + \frac{1}{2}\left[(I_x - I_z - eS_\theta^2)\Omega + (ec_{2\theta} - I_{z_2})\omega\right]\right\}eS_{2\theta}\xi_1$$

$$+ \frac{1}{\Delta}\left\{-\left[I_z - I_y - eS_\theta^2)\Omega + (e\, c_{2\theta} + I_{z_2})\omega\right](I_y + eS_\theta^2)\right.$$

$$\left. + \frac{1}{2}e^2S_\theta^2\left(\omega + \frac{1}{2}\Omega\right)\right\}\xi_2 + \frac{1}{\Delta}eS_{2\theta}\,\mathrm{mgd}\;\gamma_1 + \frac{1}{\Delta}(I_y + eS_\theta^2)\mathrm{mgd}\;\gamma_2\;, \qquad (4.47)$$

$$\dot{\xi}_2 = \frac{1}{\Delta}\left\{-(I_x - eS_\theta^2)\left[(I_x - I_z - eS_\theta^2)\Omega + (ec_{2\theta} - I_{z_2})\omega\right]\right.$$

$$\left. + \left(\omega + \frac{1}{2}\Omega\right)\frac{1}{2}e^2S_{2\theta}^2\right\}\xi_1$$

$$+ \frac{1}{\Delta}\left\{-(I_x - eS_\theta^2)\left(\omega + \frac{1}{2}\Omega\right) + \left[(I_z - I_y - eS_\theta^2)\Omega\right.\right.$$

$$\left.\left. + (ec_{2\theta} + I_{z_o})\omega\right]\right\}\frac{1}{2}eS_{2\theta}\,\xi_2$$

$$- \frac{1}{\Delta}(I_x - eS_\theta^2)\,\mathrm{mgd}\;\gamma_1 - \frac{1}{\Delta}eS_{2\theta}\,\mathrm{mgd}\;\gamma_2\;. \qquad (4.48)$$

Now, collecting the Ω and ω terms, Equations (4.44, 4.45) can be rewritten as

$$\dot{\xi} = \frac{1}{\Delta}\left[\frac{1}{2}(I_x + I_y - I_z)\Omega + \left(I_y + eS_\theta^2 + \frac{1}{2}ec_{2\theta} - \frac{1}{2}I_{z_2}\right)\omega\right]eS_{2\theta}\,\xi_1$$

$$+ \frac{1}{\Delta}\{[I_y(I_y - I_z) + (2I_y - I_z)eS_\theta^2 + e^2S_\theta^2]\,\Omega$$

$$+ [-I_z\,ec_{2\theta} - I_zI_{z_2} - I_{z_2}eS_\theta^2 + e^2S_\theta^2]\,\omega\}\,\xi_2$$

$$+ \frac{1}{\Delta}\frac{1}{2}eS_{2\theta}\,\mathrm{mgd}\;\gamma_1 + \frac{1}{\Delta}(I_y + eS_\theta^2)\mathrm{mgd}\;\gamma_2\;, \qquad (4.49)$$

$$\dot{\xi}_2 = \frac{1}{\Delta} \{[- I_x(I_x - I_z) + (2I_x - I_z)eS_\theta^2 - e^2S_\theta^2]\, \Omega$$

$$+ [- I_x\, ec_{2\theta} + I_xI_{z_2} - I_{z_2}eS_\theta^2 - e^2S_\theta^2]\, \omega\}\, \xi_1$$

$$+ \frac{1}{\Delta} \Big\{- \frac{1}{2}\, (I_x + I_y - I_z)\Omega$$

$$+ \left(- I_x + eS_\theta^2 + \frac{1}{2}\, ec_{2\theta} + \frac{1}{2}\, I_{z_2}\right) \omega\Big\}\, eS_{2\theta}\xi_2$$

$$- \frac{1}{\Delta}\, (I_x - eS_\theta^2)mgd\ \gamma_1 - \frac{1}{\Delta}\frac{1}{2}\, eS_{2\theta}\ mgd\ \gamma_2 \quad . \tag{4.50}$$

Let us introduce the following nondimensional parameters:

$$i_x = I_x/I_z \quad , \quad k = I_y/I_x \ , \quad \bar{c} = I_{z_2}/_z \ , \ \bar{e} = e/I_{z_2} \ . \tag{4.51}$$

Now, if we divide both the numerator and denominator of the right hand sides of Equations (4.49, 4.50) by I_z^2 and make use of (4.51), we obtain

$$\dot{\xi}_1 = \frac{1}{\Delta^*} \Big\{\frac{1}{2}\, [i_x(1 + k) - 1]\Omega + \Big(ki_x + \bar{c}\ \bar{e}\ S_\theta^2$$

$$+ \frac{1}{2}\, \bar{c}\ \bar{e}\ c_{2\theta} + \frac{1}{2}\, \bar{c}\Big)\, \omega\Big\}\, \bar{c}\ \bar{e}\ S_{2\theta}\ \xi_1$$

$$+ \frac{1}{\Delta^*}\, \{[ki_x(ki_x - 1) + (2ki_x - 1)\bar{c}\ \bar{e}\ S_\theta^2 + \bar{c}\ \bar{e}^2S_\theta^2]\, \Omega$$

$$+ [- ki_x\ \bar{c}\ \bar{e}\ c_{2\theta} - ki_x\ \bar{c} - \bar{c}^2\bar{e}\ S_\theta^2 + \bar{c}^2\bar{e}^2S_\theta^2]\, \omega\Big\}\, \xi_2$$

$$+ \frac{1}{\Delta^*}\frac{1}{2}\, \bar{c}\ \bar{e}\ S_{2\theta}\, \frac{mgd}{I_z}\, \eta_1 + \frac{1}{\Delta^*}\, (ki_x + \bar{c}\ \bar{e}\ S_\theta^2)\, \frac{mgd}{I_z}\, \eta_2 \quad , \tag{4.52}$$

$$\dot{\xi}_2 = \frac{1}{\Delta^*} \{[- i_x(i_x - 1) + (2i_x - 1)\bar{c}\ \bar{e}\ s_\theta^2 - \bar{c}^2\bar{e}^2 s_\theta^2]\Omega$$

$$+ [- i_x\bar{c}\ \bar{e}\ c_{2\theta} + i_x\bar{c} - \bar{c}^2\bar{e}s_\theta^2 - \bar{c}^2\bar{e}^2 s_\theta^2]\ \omega\}\ \xi_1$$

$$+ \frac{1}{\Delta^*} \left\{- \frac{1}{2}\ (i_x(1 + k) - 1]\ \Omega + \left(- i_x + \bar{c}\ \bar{e}s_\theta^2 + \frac{1}{2}\ \bar{c}\ \bar{e}\ c_{2\theta}\right.\right.$$

$$\left.\left. + \frac{1}{2}\ \bar{c}\right)\ \omega\right\}\ \bar{c}\ \bar{e}s_{2\theta}\ \xi_2 - \frac{1}{\Delta^*}\ (i_x - \bar{c}\ \bar{e}\ s_\theta^2)\ \frac{mgd}{I_z}\ \eta_1$$

$$- \frac{1}{\Delta^*}\ \frac{1}{2}\ \bar{c}\ \bar{e}\ s_{2\theta}\ \frac{mgd}{I_z}\ \eta_2 \quad , \tag{4.53}$$

where

$$\Delta^* = \frac{\Delta}{(I_z)^2} = ki_x^2 + [i_x(1 - k) - \bar{c}\ \bar{e}]\bar{c}\ \bar{e}\ s_\theta^2 \ . \tag{4.54}$$

Now let us define a nondimensional time parameter

$$\tau^* = \omega t \qquad (\omega \neq 0) \quad , \tag{4.55}$$

such that the differential operators $\frac{d}{dt}$ and $\frac{d}{d\tau^*}$ are related by

$$\frac{d}{dt} = \omega \frac{d}{d\tau^*} \quad . \tag{4.56}$$

Here we also introduce nondimensional functions $\bar{\xi}_1$ and $\bar{\xi}_2$ such that

$$\bar{\xi}_1 = \xi_1/\Omega \quad \text{and} \quad \bar{\xi}_2 = \xi_2/\Omega \ , \tag{4.57}$$

where we are assuming $\Omega \neq 0$.

Substituting Equations (4.55, 4.57) into Equations (4.52, 4.53) and dividing by $\omega\,\Omega$ yields the following pair of differential equations:

$$\bar{\xi}_1' = \frac{1}{\Delta^*}\left\{\frac{1}{2}\left[i_x(1+k)-1\right]\frac{\Omega}{\omega} + \left(ki_x + \bar{c}\;\bar{e}S_\theta^2 + \frac{1}{2}\bar{c}\;\bar{e}\;c_{2\theta} + \frac{1}{2}\bar{c}\right)\right\}\bar{c}\;\bar{e}S_{2\theta}\bar{\xi}_1$$

$$+ \frac{1}{\Delta^*}\left\{\left[ki_x(ki_x-1) + (2ki_x-1)\bar{c}\;\bar{e}S_\theta^2 + \bar{c}^2\bar{e}^2S_\theta^2\right]\frac{\Omega}{\omega}\right.$$

$$\left. + \left[-ki_x\,\bar{c}\;\bar{e}\;c_{2\theta} - ki_x\bar{c} - \bar{c}^2\bar{e}S_\theta^2 + \bar{c}^2\bar{e}^2S_\theta^2\right]\right\}\bar{\xi}_2$$

$$+ \frac{1}{\Delta^*}\frac{1}{2}\bar{c}\;\bar{e}S_{2\theta}\frac{mgd}{I_z\Omega\omega}\eta_1 + \frac{1}{\Delta^*}(ki_x + \bar{c}\;\bar{e}S_\theta^2)\frac{mgd}{I_z\Omega\omega}\eta_2 \quad , \tag{4.58}$$

$$\bar{\xi}_2' = \frac{1}{\Delta^*}\left\{\left[-i_x(i_x-1) + (2i_x-1)\bar{c}\;\bar{e}S_\theta^2 - \bar{c}^2\bar{e}^2S_\theta^2\right]\frac{\Omega}{\omega}\right.$$

$$\left. + \left[-i_x\,\bar{c}\;\bar{e}\;c_{2\theta} + i_x\,\bar{c} - \bar{c}^2\bar{e}S_\theta^2 - \bar{c}^2\bar{e}^2S_\theta^2\right]\right\}\bar{\xi}_1$$

$$+ \frac{1}{\Delta^*}\left\{-\frac{1}{2}\left[i_x(1+k)-1\right]\frac{\Omega}{\omega}\right.$$

$$\left. + \left(-i_x + \bar{c}\;\bar{e}S_\theta^2 + \frac{1}{2}\bar{c}\;\bar{e}c_{2\theta} + \frac{1}{2}\bar{c}\right)\right\}\bar{c}\;\bar{e}S_{2\theta}\,\bar{\xi}_2$$

$$- \frac{1}{\Delta^*}(i_x - \bar{c}\;\bar{e}S_\theta^2)\frac{mgd}{I_z\Omega\omega}\eta_1 - \frac{1}{\Delta^*}\frac{1}{2}\bar{c}\;\bar{e}S_{2\theta}\frac{mgd}{I_z\Omega\omega}\eta_2 \quad , \tag{4.59}$$

where

$$(\quad)' = \frac{d}{d\tau^*} \quad .$$

Likewise if we substitute Equation (4.57) into Equations (4.42, 4.43) and divide the results by $\omega\,\Omega$ we obtain

$$\eta_1' = -\frac{\Omega}{\omega}\bar{\xi}_2 + \frac{\Omega}{\omega}\eta_2 \ , \tag{4.60}$$

$$\eta_2' = \frac{\Omega}{\omega}\bar{\xi}_1 - \frac{\Omega}{\omega}\eta_1 \ , \tag{4.61}$$

where again

$$(\quad)' = \frac{d}{d\tau^*} \ .$$

Let us now introduce a nondimensional spin parameter S^* and a nondimensional relative spin parameter β which are defined by

$$S^* = \frac{I_z \Omega^2}{mg|d|} \ , \tag{4.62}$$

and

$$\beta = \frac{I_{z_2}\omega}{\sqrt{I_z mg|d|}} \ . \tag{4.63}$$

Thus the terms $\frac{\Omega}{\omega}$ and $\frac{mgd}{I_z\omega\Omega}$ become

$$\frac{\Omega}{\omega} = \frac{\bar{c}\sqrt{S^*}}{\beta} \tag{4.64}$$

and

$$\frac{mgd}{I_z\Omega\omega} = \operatorname{sgn}(d)\,\frac{\bar{c}}{\beta\sqrt{S^*}} \ . \tag{4.65}$$

Substituting Equations (4.64, 4.65) into Equations (4.58, 4.61) yields:

$$\bar{\xi}_1' = \frac{1}{\Delta^*}\left\{\frac{1}{2}\bar{c}[i_x(1+k)-1]\frac{\sqrt{S^*}}{\beta}\right.$$

$$\left.+\left(ki_x+\bar{c}\ \bar{e}S_\theta^2+\frac{1}{2}\bar{c}\ \bar{e}c_{2\theta}+\frac{1}{2}\bar{c}\right)\right\}\bar{c}\ \bar{e}S_{2\theta}\ \bar{\xi}_1$$

$$+\frac{1}{\Delta^*}\left\{[ki_x(ki_x-1)+(2ki_x-1)\bar{c}\ \bar{e}S_\theta^2+\bar{c}^2\bar{e}^2S_\theta^2]\frac{\bar{c}\sqrt{S^*}}{\beta}\right.$$

$$\left.+[-ki_x\bar{c}\ \bar{e}\ c_{2\theta}-ki_x\bar{c}-\bar{c}^2\bar{e}S_\theta^2+\bar{c}^2\bar{e}^2S_\theta^2]\right\}\bar{\xi}_2$$

$$+\frac{1}{\Delta^*}\frac{1}{2}\bar{c}^2\bar{e}S_{2\theta}\ \mathrm{sgn}(d)\frac{1}{\beta\sqrt{S^*}}\eta_1$$

$$+\frac{1}{\Delta^*}(ki_x+\bar{c}\ \bar{e}S_\theta^2)\mathrm{sgn}(d)\frac{\bar{c}}{\beta\sqrt{S^*}}\eta_2\quad, \tag{4.66}$$

$$\bar{\xi}_2' = \frac{1}{\Delta^*}\left\{[-i_x(i_x-1)+(2i_x-1)\bar{c}\ \bar{e}S_\theta^2-\bar{c}\ \bar{e}^2S_\theta^2]\frac{\bar{c}\sqrt{S^*}}{\beta}\right.$$

$$\left.+[-i_x\bar{c}\ \bar{e}\ c_{2\theta}+i_x\bar{c}-\bar{c}^2\bar{e}S_\theta^2-\bar{c}^2\bar{e}^2S_\theta^2]\right\}\bar{\xi}_1$$

$$+\frac{1}{\Delta^*}\left\{-\frac{1}{2}\bar{c}[i_x(1+k)-1]\frac{\sqrt{S^*}}{\beta}\right.$$

$$\left.+\left(-i_x+\bar{c}\ \bar{e}S_\theta^2+\frac{1}{2}\bar{c}\ c_{2\theta}+\frac{1}{2}\bar{c}\right)\right\}\bar{c}\ \bar{e}S_{2\theta}\ \bar{\xi}_2$$

$$-\frac{1}{\Delta^*}(i_x-\bar{c}\ \bar{e}S_\theta^2)\mathrm{sgn}(d)\frac{\bar{c}}{\beta\sqrt{S^*}}\gamma_1-\frac{1}{\Delta^*}\frac{1}{2}\bar{c}^2\bar{e}S_{2\theta}\mathrm{sgn}(d)\frac{1}{\beta\sqrt{S^*}}\gamma_2, \tag{4.67}$$

$$\eta_1' = \frac{\bar{c}\sqrt{S^*}}{\beta}\,\bar{\xi}_2 + \frac{\bar{c}\sqrt{S^*}}{\beta}\,\eta_2 \;, \tag{4.68}$$

$$\eta_2' = \frac{\bar{c}\sqrt{S^*}}{\beta}\,\bar{\xi}_1 - \frac{\bar{c}\sqrt{S^*}}{\beta}\,\eta_1 \;. \tag{4.69}$$

In order to make a comparison with the stability-instability results for a gyrostat we set

$$S = \mathrm{sgn}(d) \cdot S^* \quad , \tag{4.70}$$

as was done in Chapter III.

Thus, Equations (4.66-4.69) given the stability of linearized system for the present problem of the extended gyrostat.

If we let $(x_1,x_2,x_3,x_4) = (\bar{\xi}_1,\bar{\xi}_2,\eta_1,\eta_2)$, then Equations (4.66-4.69) can be put into the matrix form

$$\{X\} = [A(\tau^*)]\{x\} \;, \tag{4.71}$$

where

$$\{X\} = \mathrm{col.}\;(\bar{\xi}_1,\; \bar{\xi}_2,\; \eta_1,\; \eta_2) \quad , \tag{4.72}$$

and in which, by expressing S_θ^2 as a function of $2\theta_3$ it is clear that $[A(\tau^*)]$ is a periodic matrix of period π. Therefore, the linearized system represented by the matrix equation (4.71) is of the required form for investigation for instability and infinitesimal stability by Floquet's theory, which was described in Chapter II.

The parameters involved in this investigation are $S, i_x, k, \beta, \bar{e}$ and $\bar{c}$. These particular parameters were chosen in order to make a comparison with the results of the gyrostat of Chapter III; but in the present case of the extended gyrostat, we have introduced two new parameters, an inertia imbalance eccentricity $\bar{e}$ and an inertia ratio $\bar{c}$ which are given in Equation (4.51). Again we take $k = 0.7$ and choose the values of β to be -1.0, 1.0, and 2.0. The case $\beta = 0.0$ is omitted because it corresponds to the special case of an arbitrary rigid body with a fixed point which was investigated for its stability in Chapter II.

In the present case, $0 \le I_{z_2} < I_z$ and $\bar{c} = I_{z_2}/I_z$; hence, it is obvious that

$$0 \le \bar{C} < 1 \quad . \tag{4.73}$$

Also, since $(I_{x_2}, I_{y_2}, I_{z_2})$ are the principal moments of inertia of body B_2 with respect to its mass center, it can easily be shown that[1]

$$-1 \le \bar{e} \le 1 \quad , \tag{4.74}$$

where $\bar{e} = \dfrac{I_{x_2} - I_{y_2}}{I_{z_2}}$. Thus, without loss in generality we restrict our stability analysis to the case $0 \le \bar{e} \le 1$, (i.e., $I_{x_2} \ge I_{y_2}$).

To illustrate the effects of the new parameters $\bar{e}$ and $\bar{c}$ on the regions of infinitesimal stability and instability obtained in Chapter III for the gyrostat, for each β we choose values of $\bar{c}$ to be 0.1 and 0.3 and for each value of $\bar{c}$ we take $\bar{e}$ to be 0.1, 0.6, and 0.9. Choosing S as the ordinate and i_x as the abscissa, Figures 12 through 29 are the Floquet results for the values of β, $\bar{e}$ and $\bar{c}$ that we have indicated, where an O is used to indicate (infinitesimal) stability and an x is

[1]The case $\bar{e} = \pm 1$ is included to allow consideration of the limiting, but physically impossible, cases in which all the mass lies in a plane.

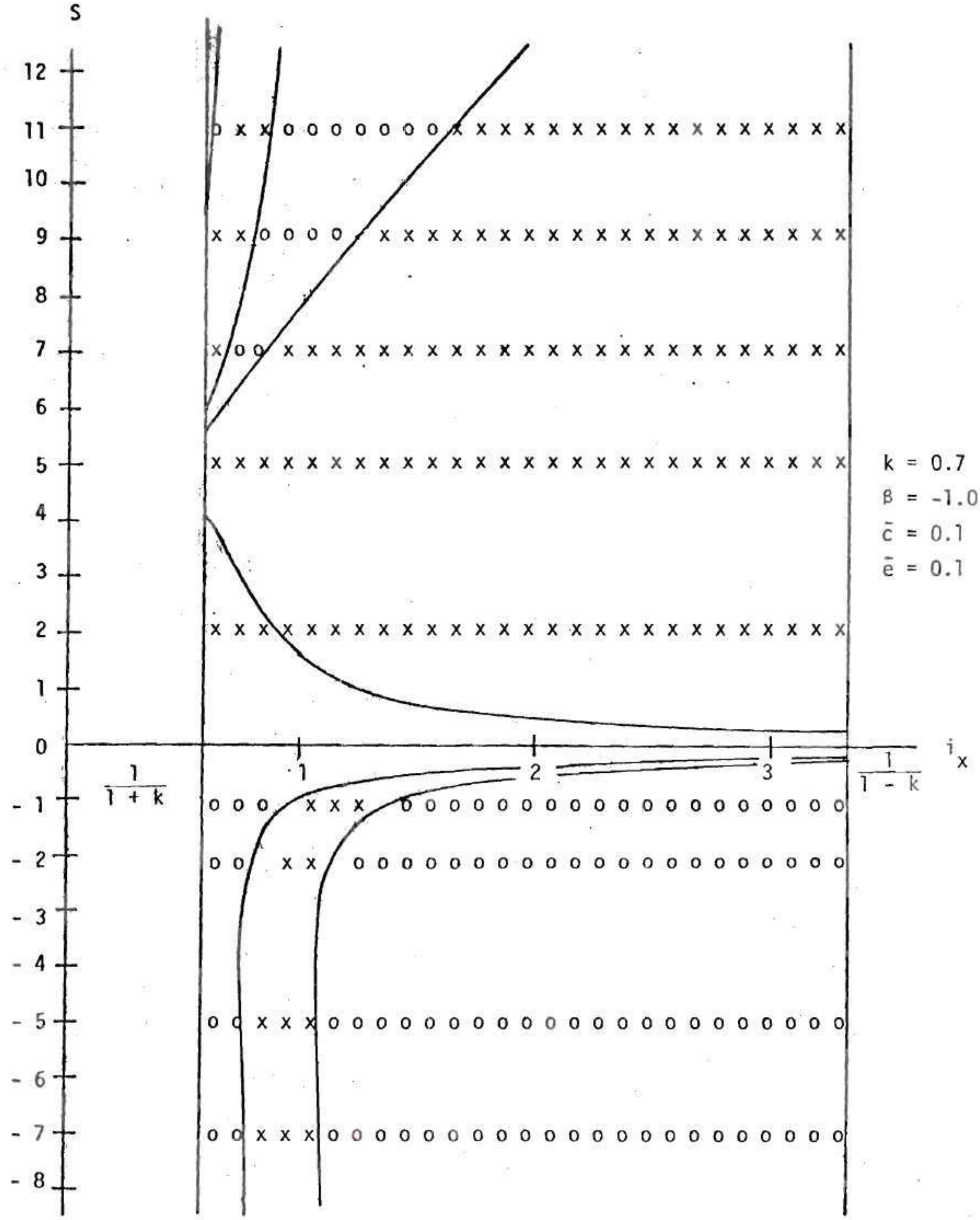

Figure 12. Extended Gyrostat.

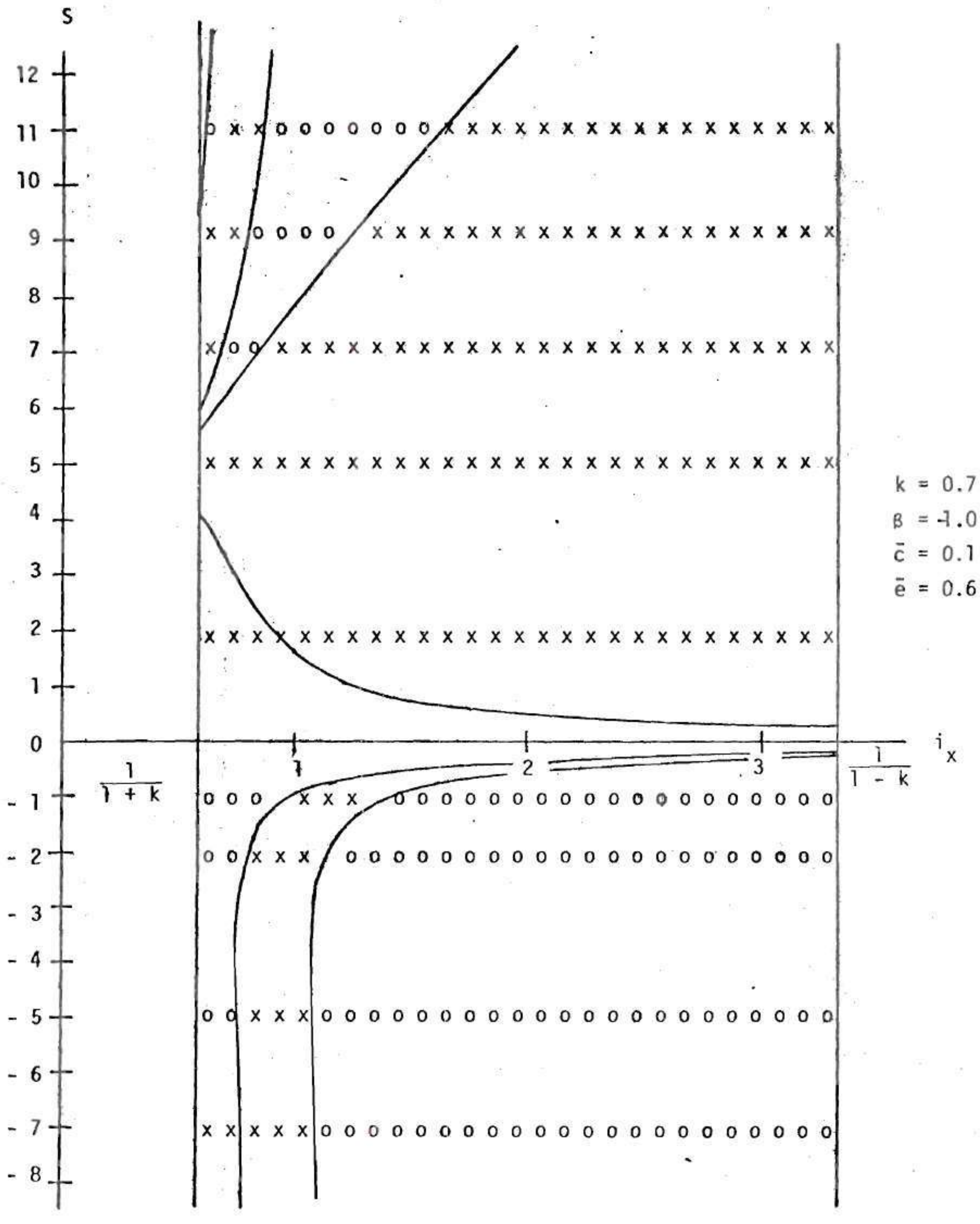

Figure 13. Extended Gyrostat.

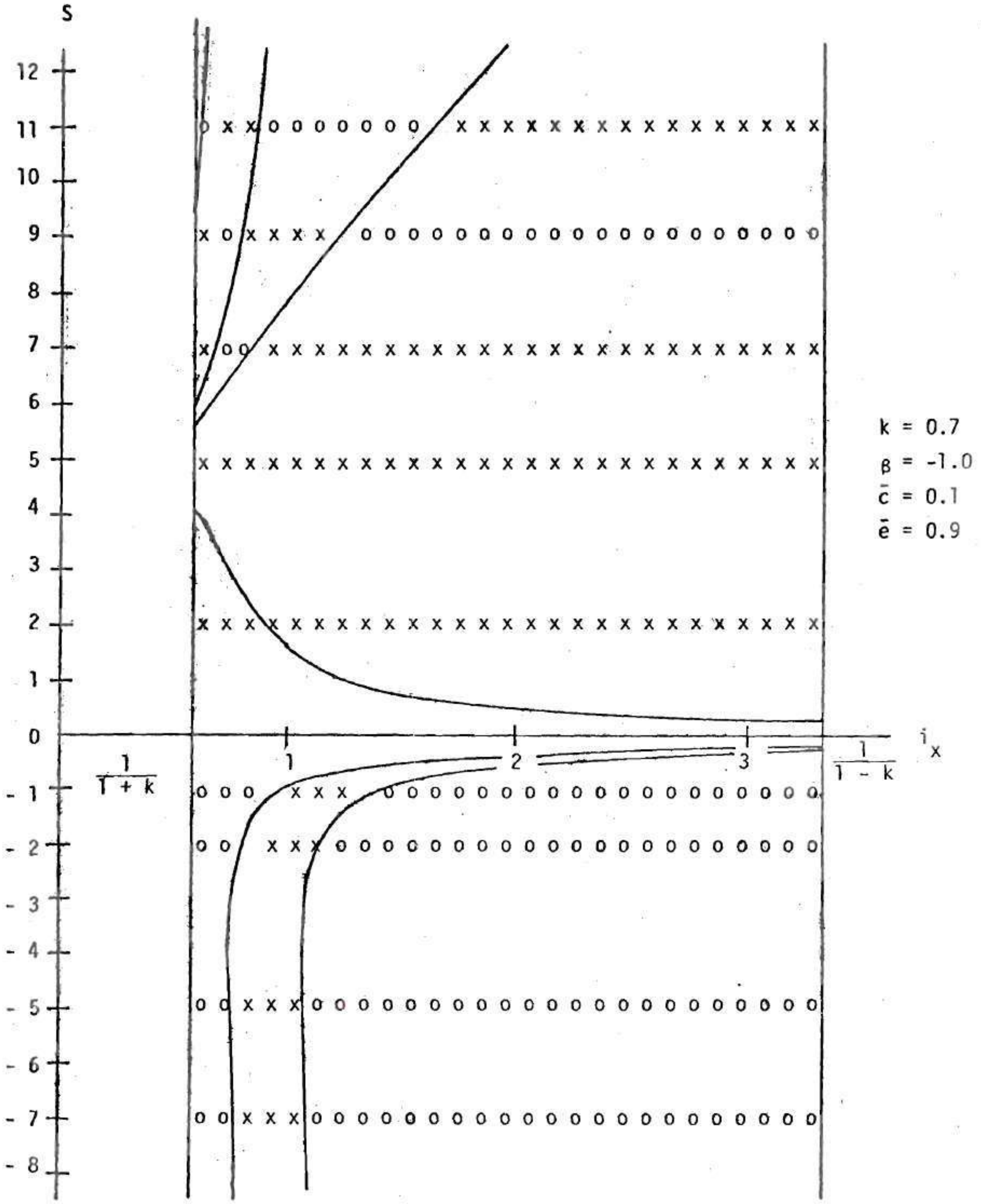

Figure 14. Extended Gyrostat.

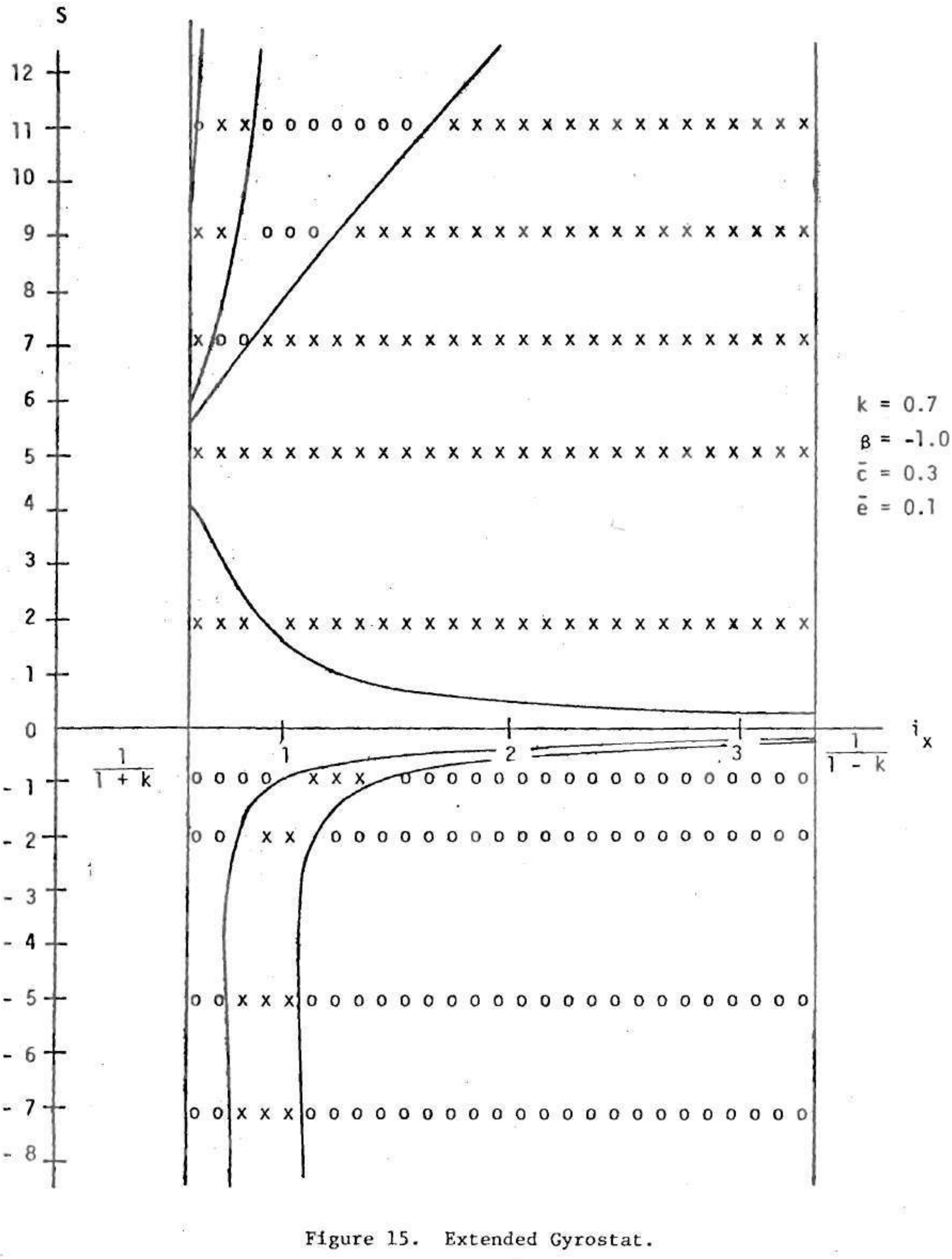

Figure 15. Extended Gyrostat.

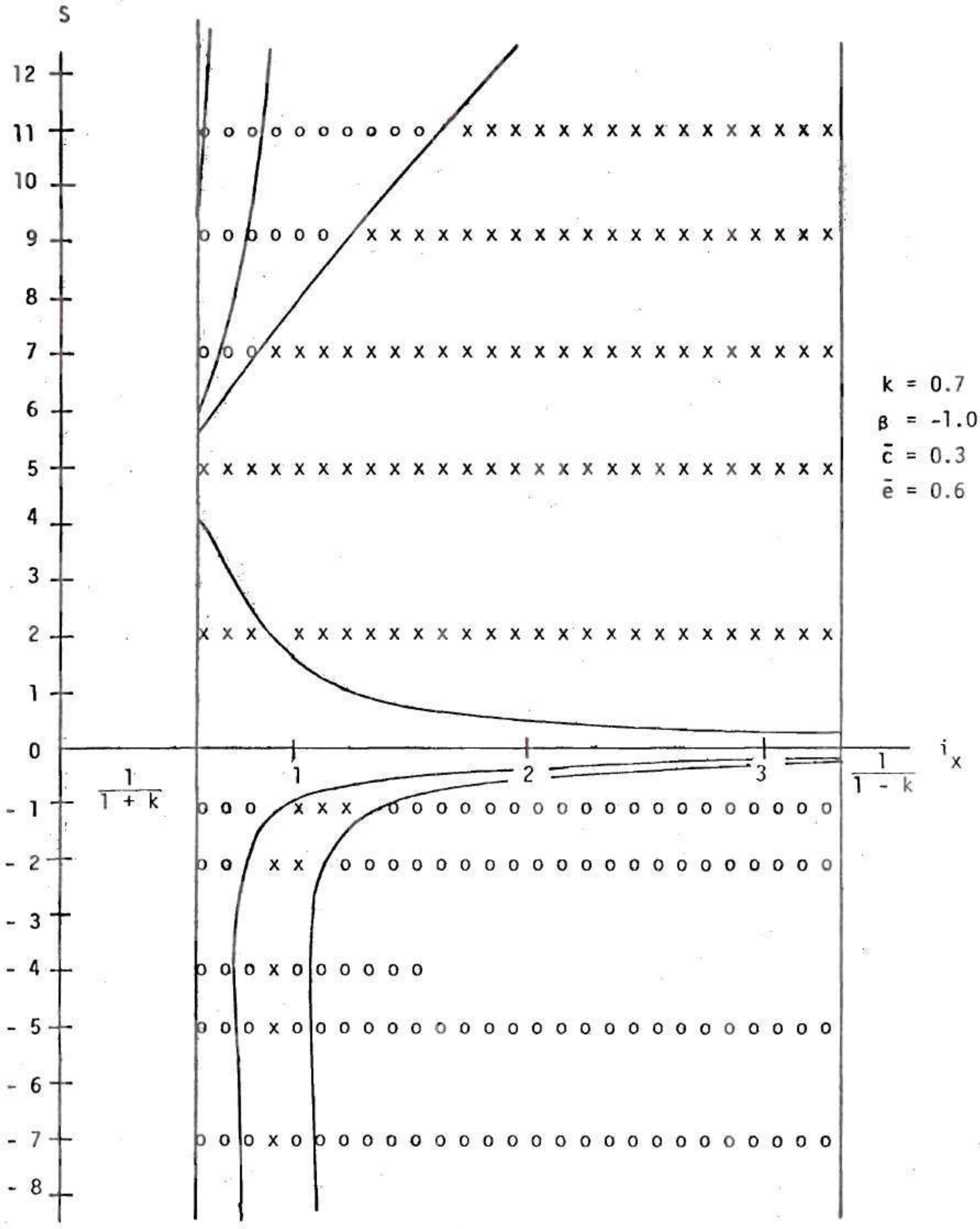

Figure 16. Extended Gryostat.

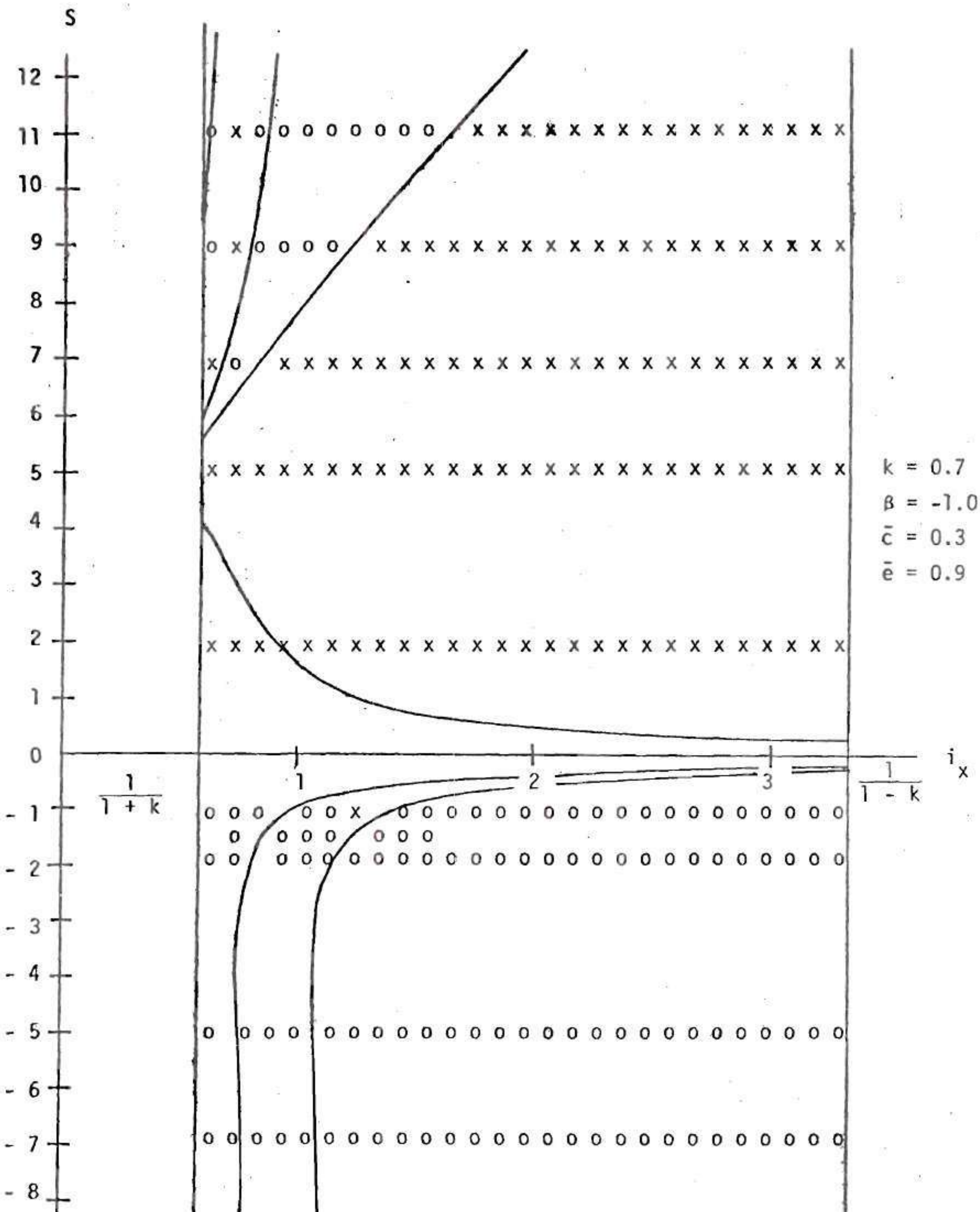

Figure 17. Extended Gyrostat.

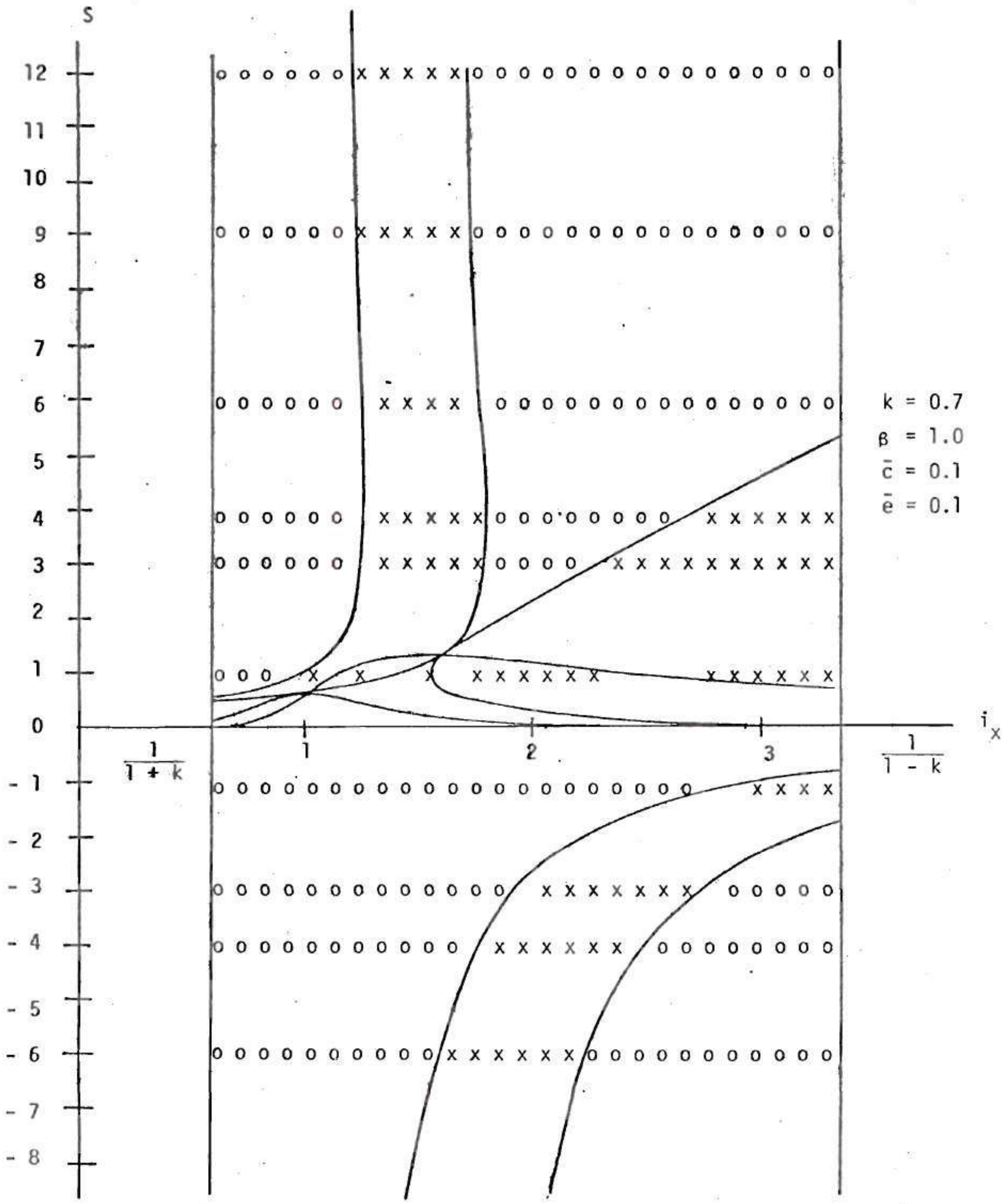

Figure 18. Extended Gyrostat.

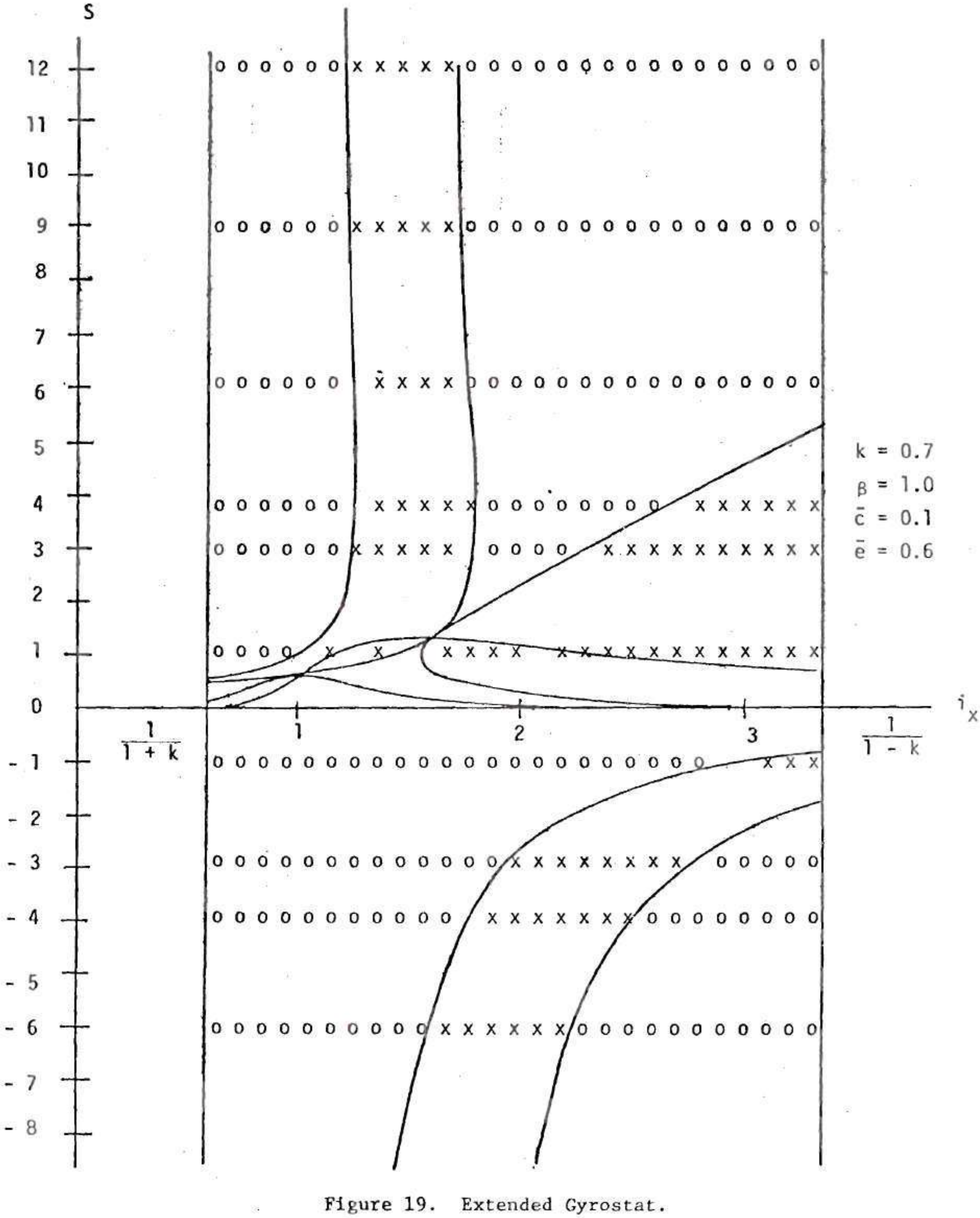

Figure 19. Extended Gyrostat.

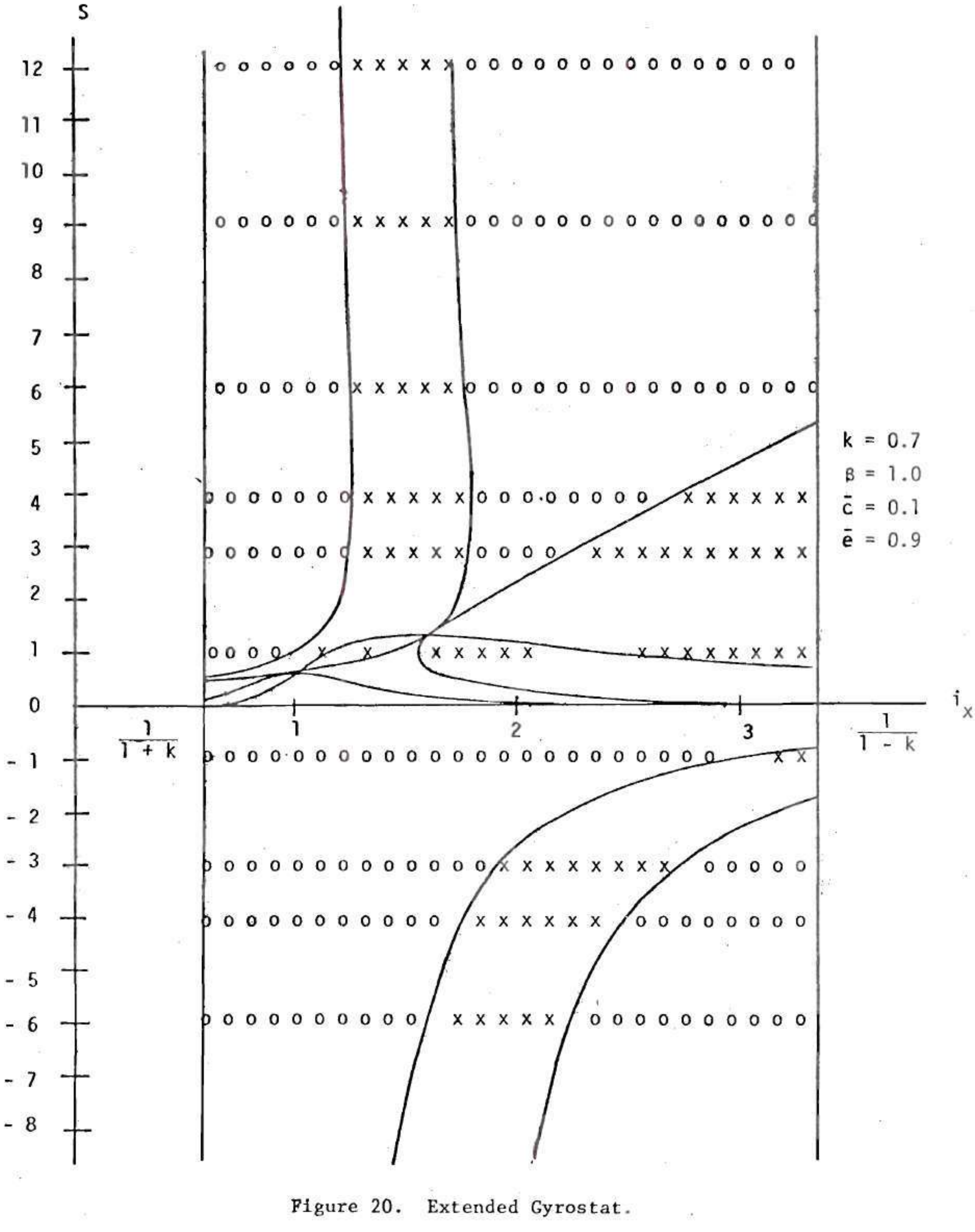

Figure 20. Extended Gyrostat.

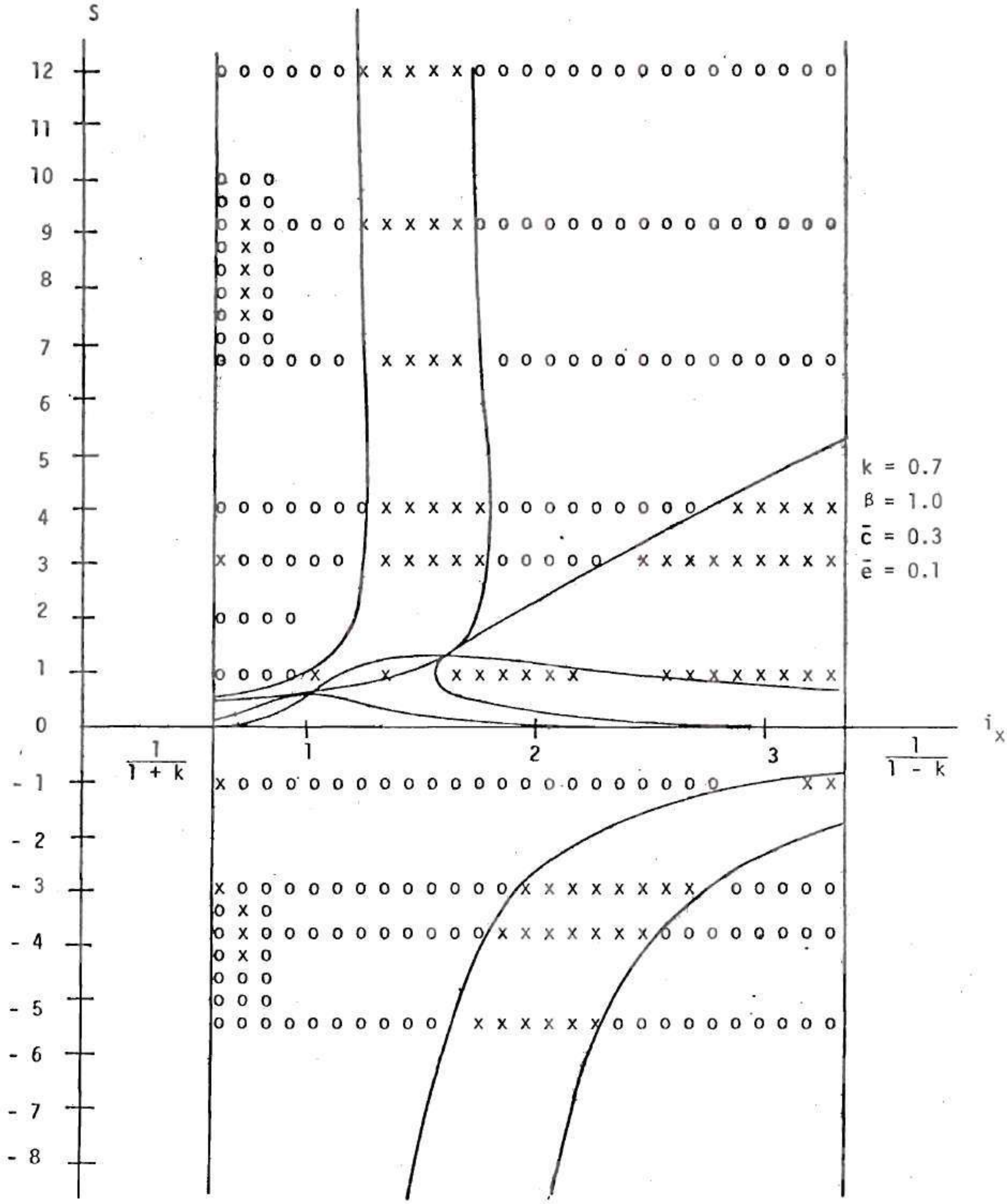

Figure 21. Extended Gyrostat.

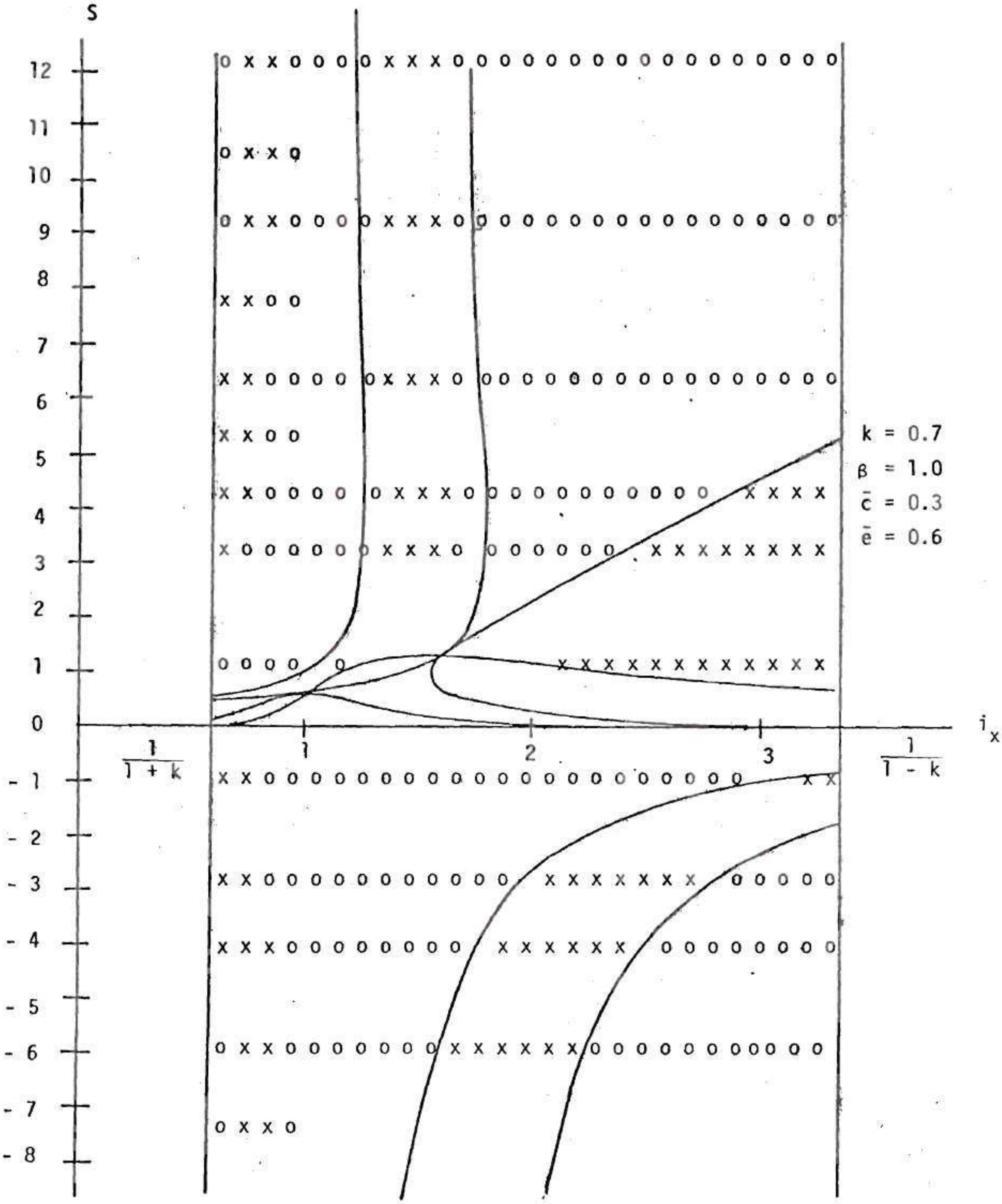

Figure 22. Extended Gyrostat.

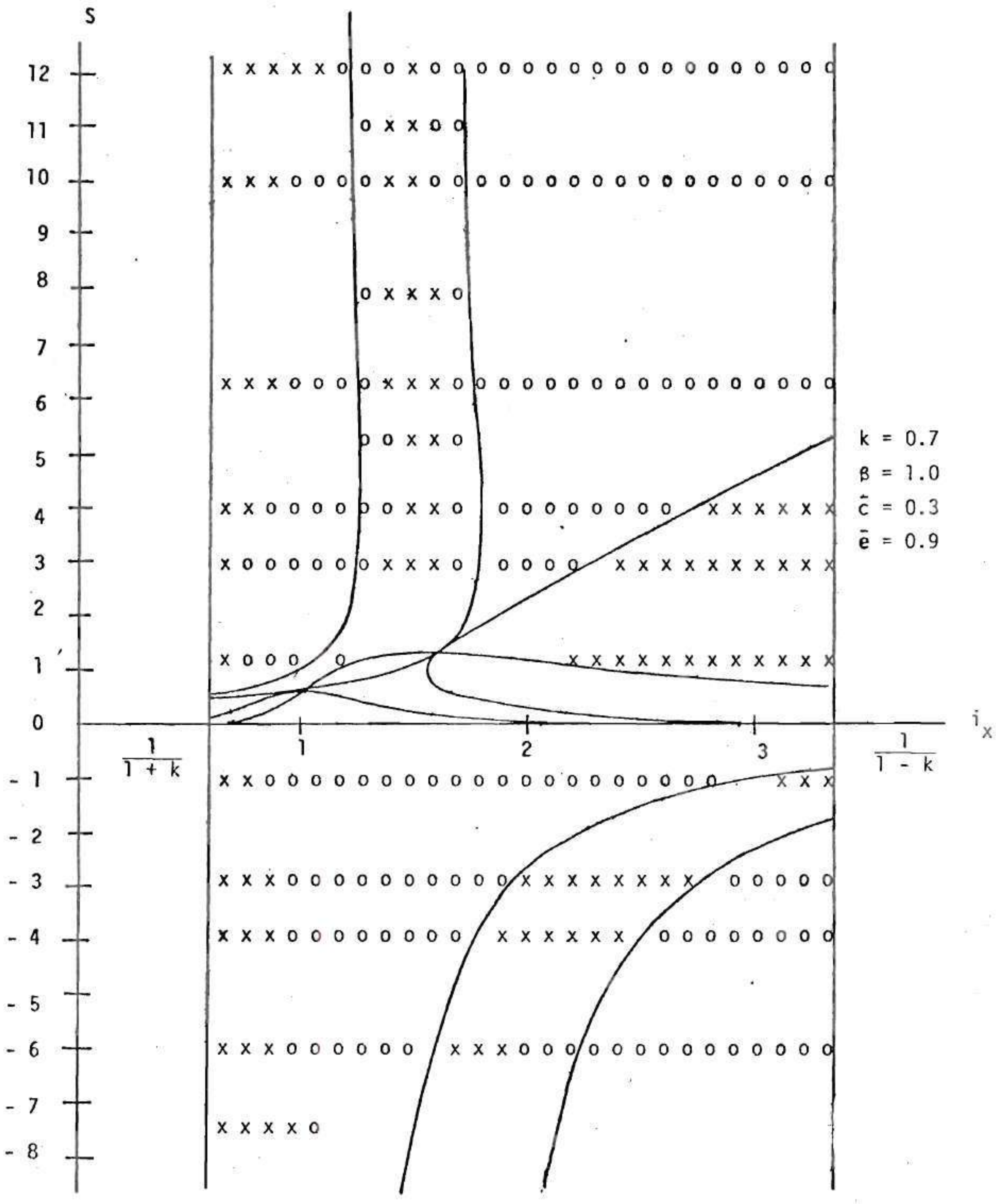

Figure 23. Extended Gyrostat.

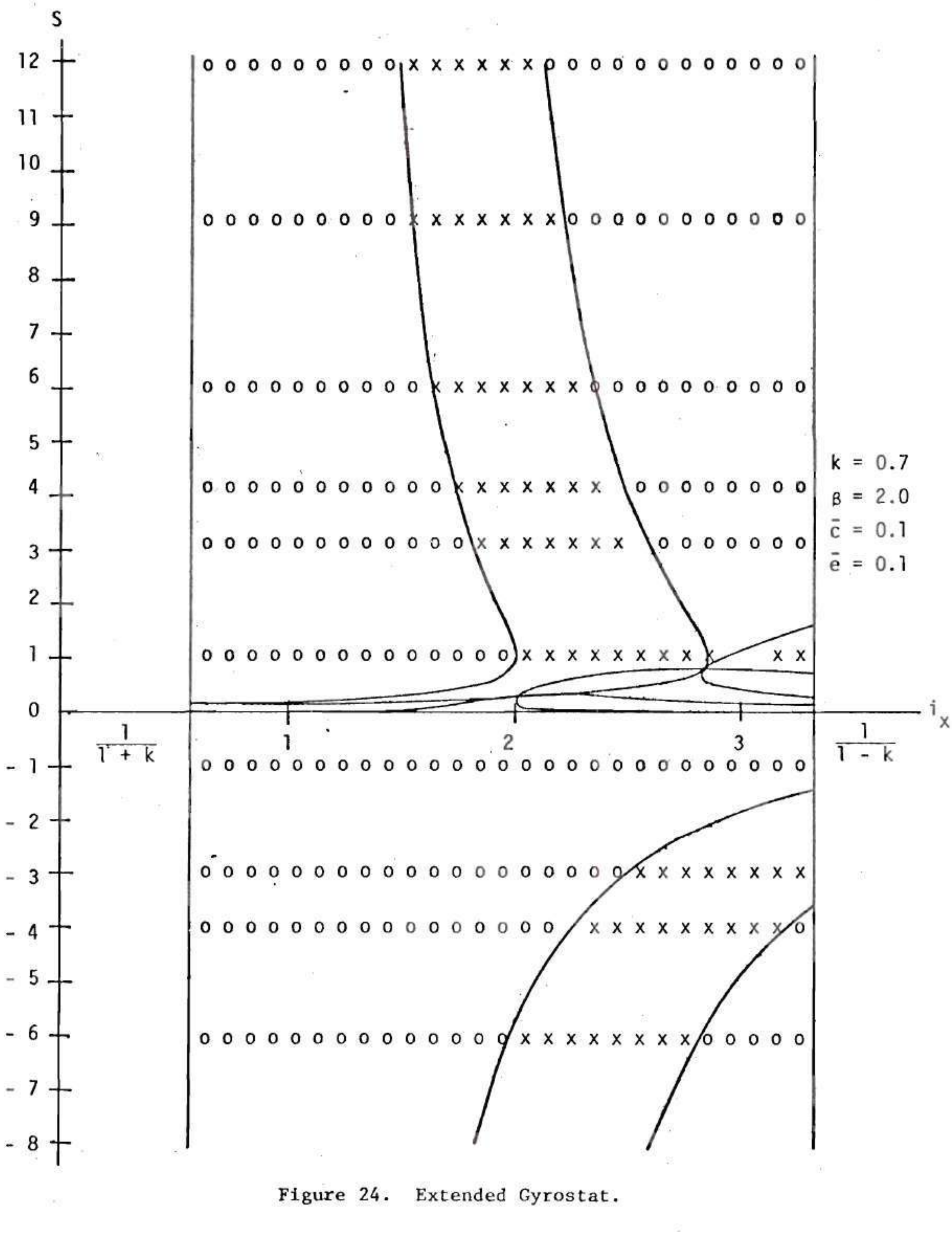

Figure 24. Extended Gyrostat.

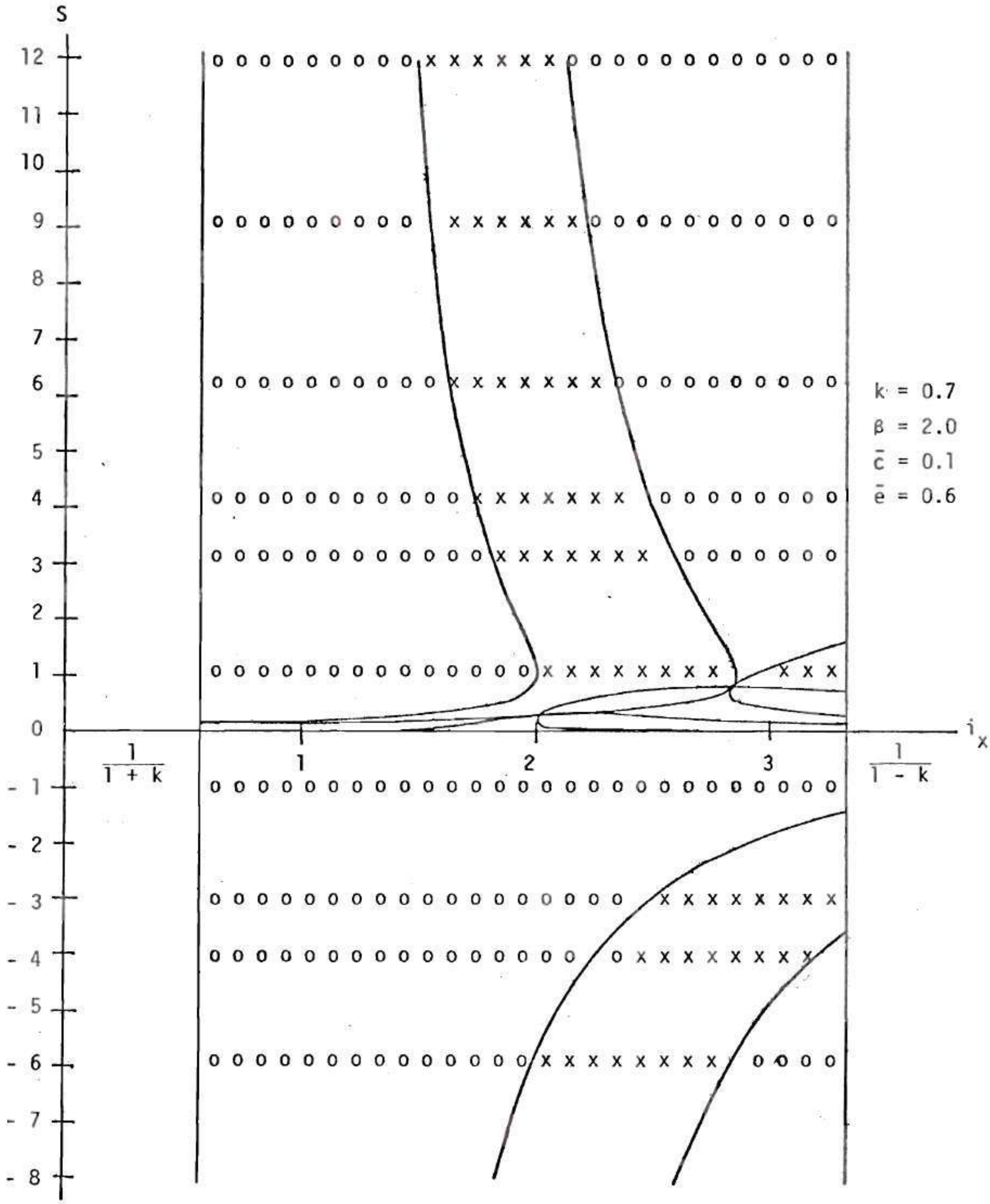

Figure 25. Extended Gyrostat.

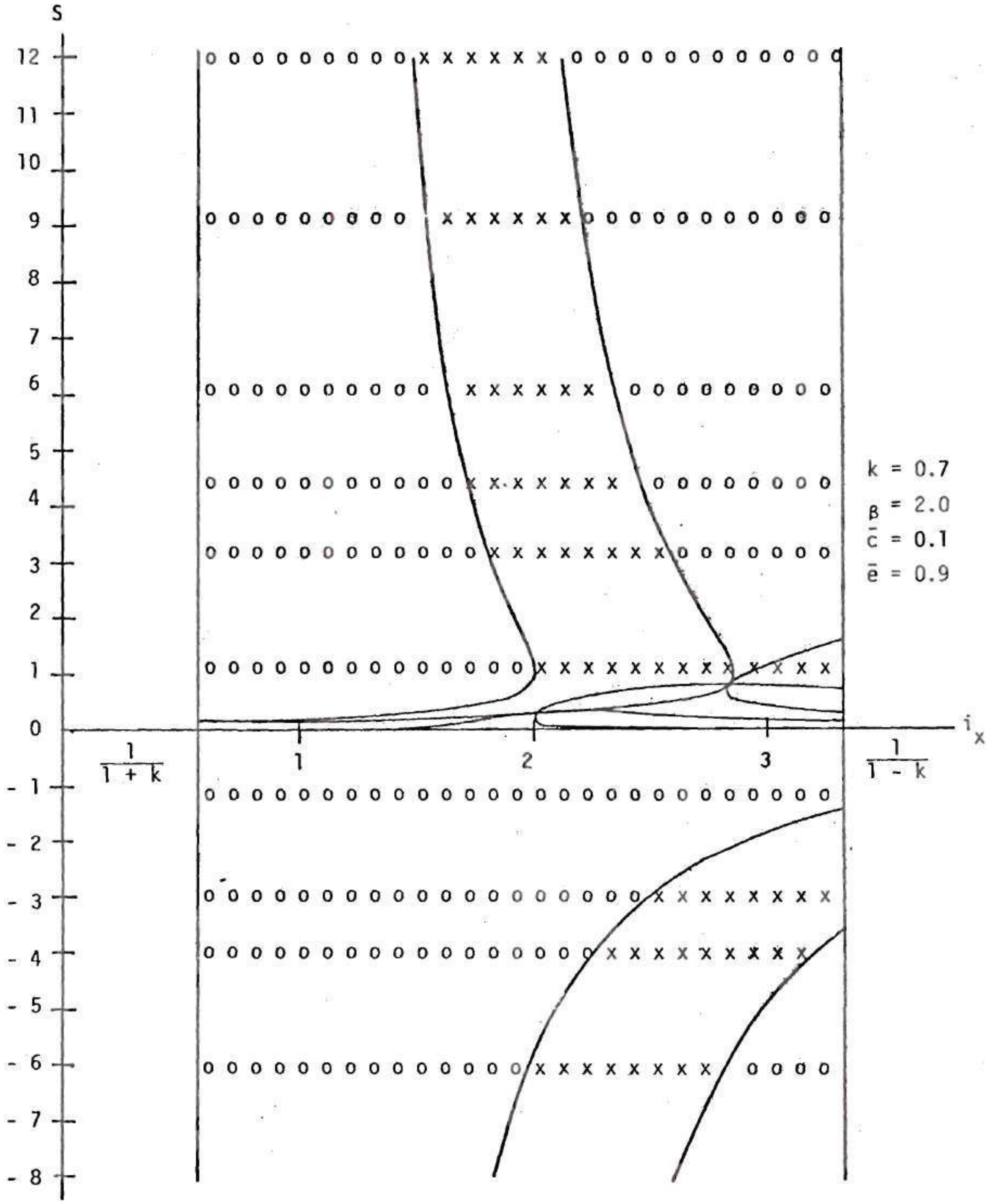

Figure 26. Extended Gyrostat.

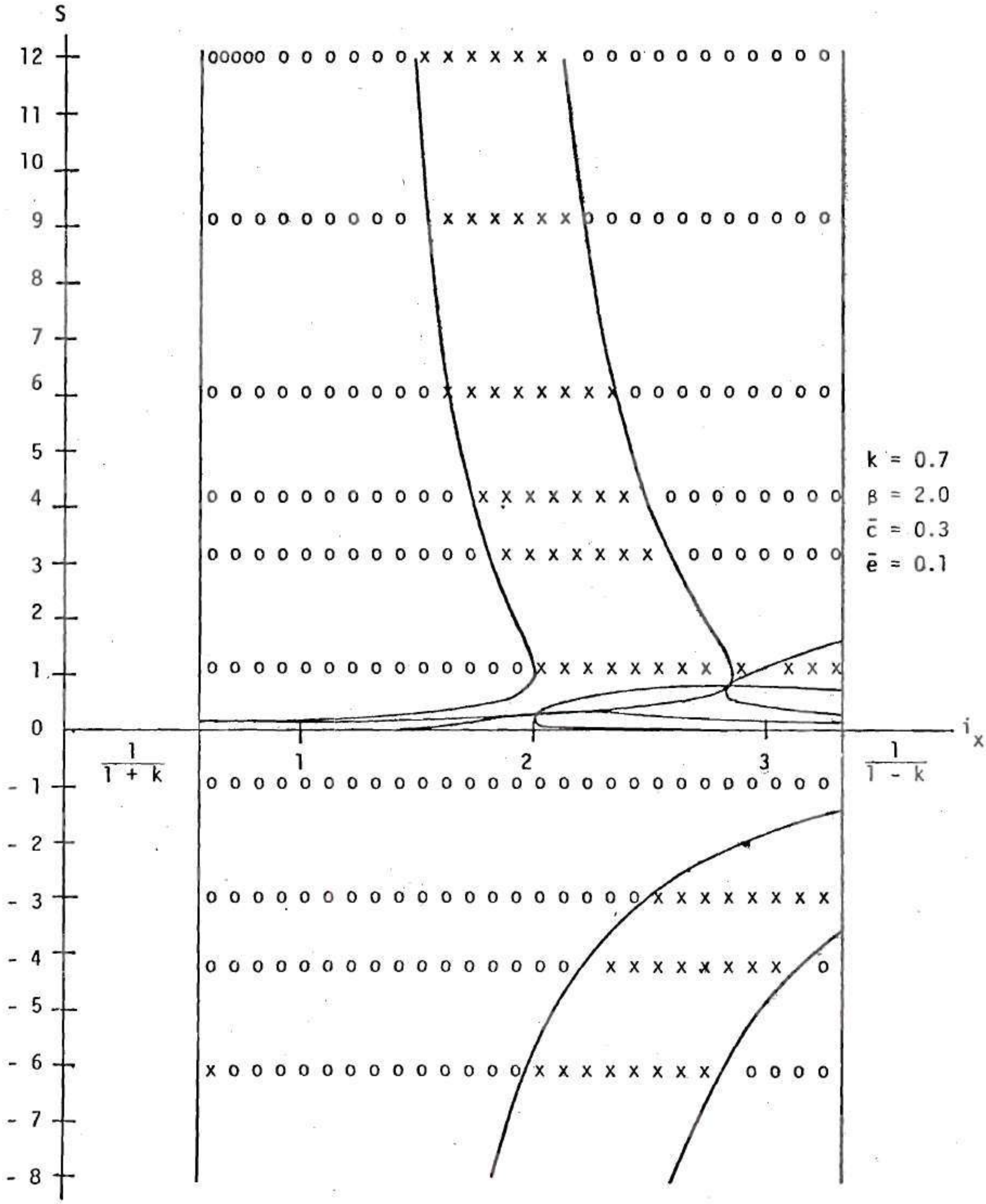

Figure 27. Extended Gyrostat.

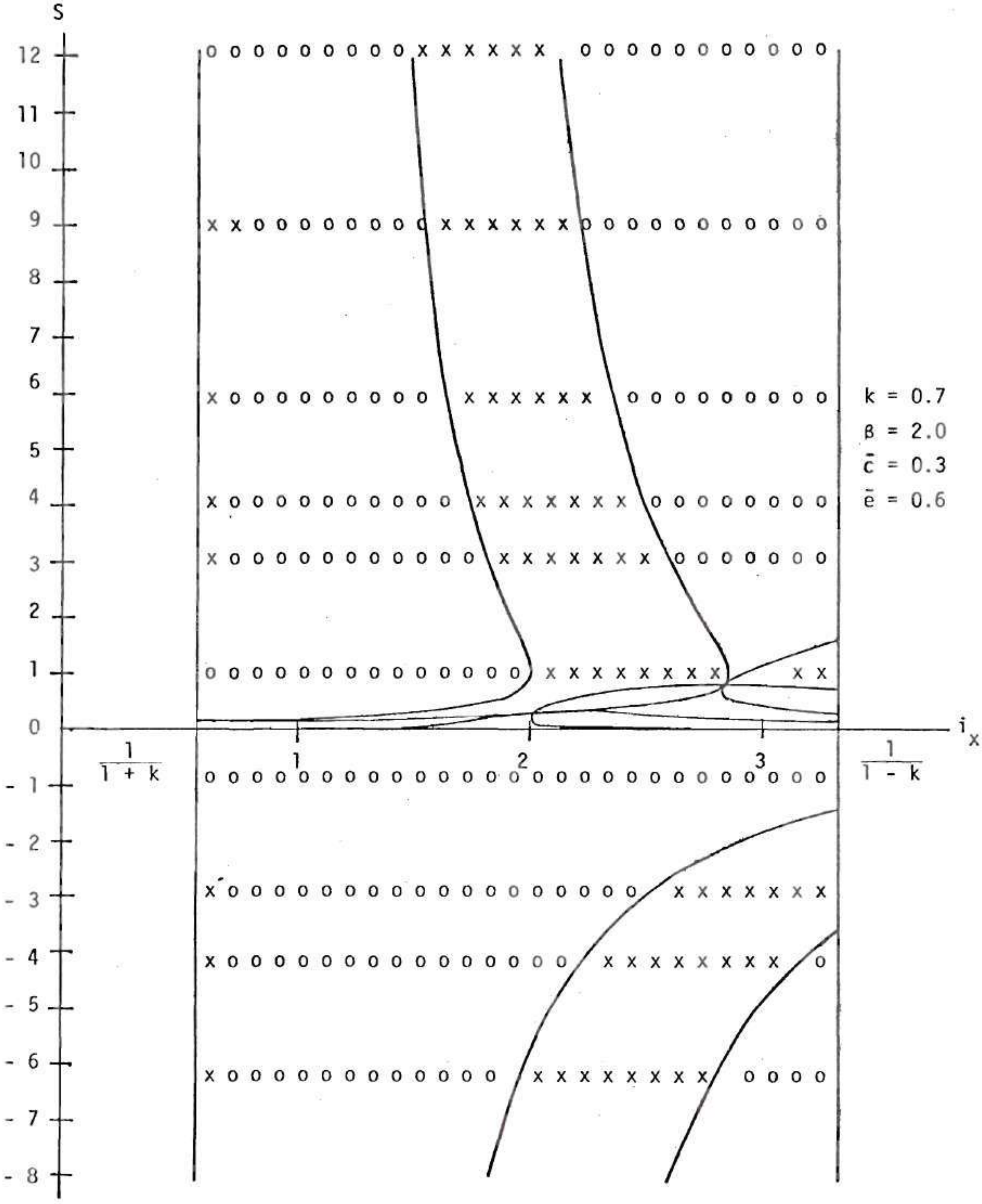

Figure 28. Extended Gyrostat.

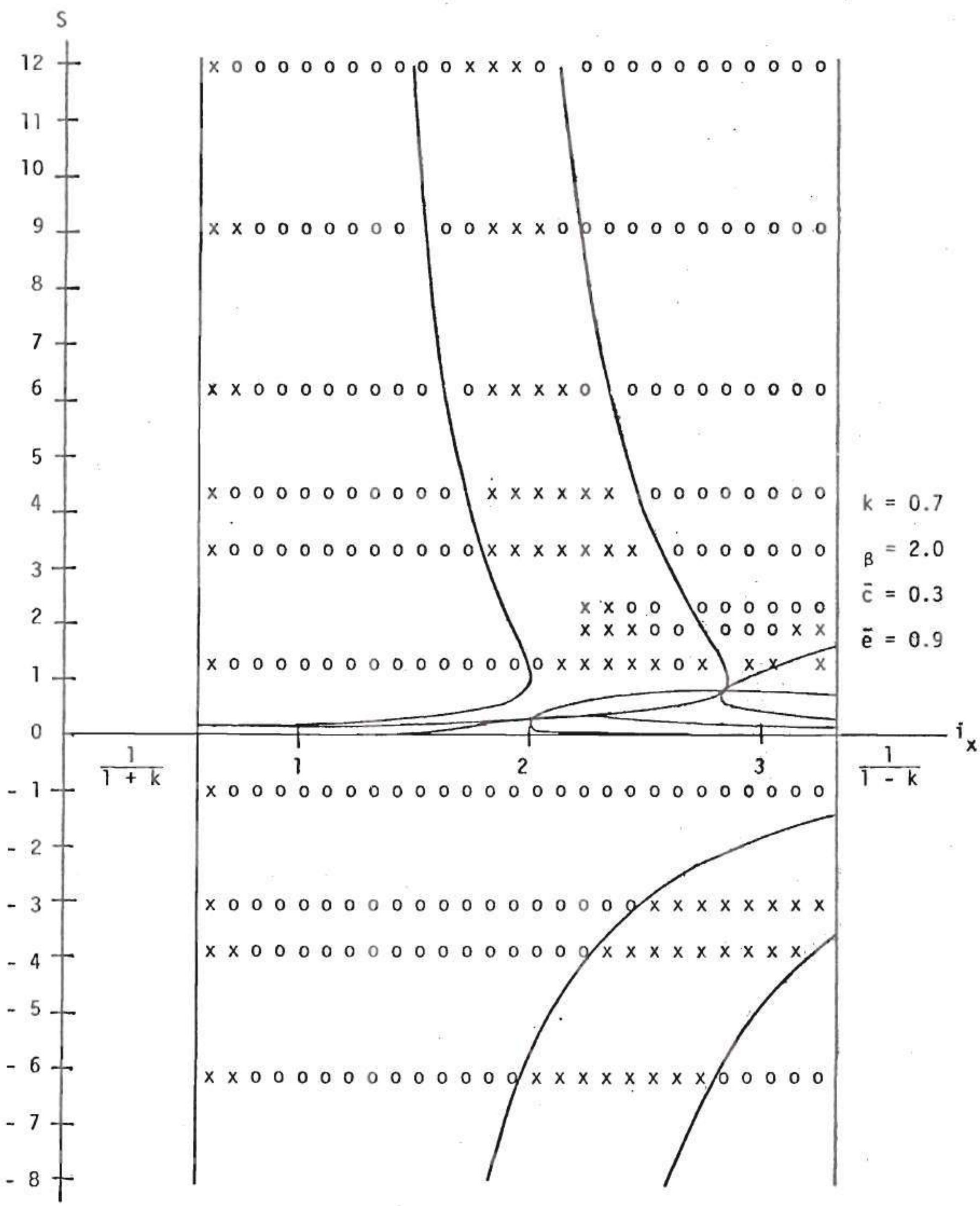

Figure 29. Extended Gyrostat.

used to indicate instability. The curves which are presented in each figure correspond to the cases where $\bar{e} = 0$, in which we obtain a gyrostat. These curves were obtained in Chapter III.

For all of the cases considered here where $\bar{c} = 0.1$, the stability-instability regions for the extended gyrostat (i.e. $\bar{e} \neq 0$) seem to be almost identical to that of the gyrostat represented by the curves. We note that here that a few points that lie just inside the unstable smokestack region in the case of the gyrostat are now indicated to be infinitesimally stable.

For $\bar{c} = 0.3$, we note a considerable change in the indicated stability-instability regions represented by the curves. Surprisingly, in the case $\beta = -1.0$, the addition of inertial imbalance of B_2 has enhanced stability; many of the points in the unstable zones of the gyrostat have now become at least infinitesimally stable for the extended gyrostat. This is very clear in Figure 17. In the cases of $\beta = 1$ and 2, some more equally important results are obtained. It is indicated in Figures 21 through 23, 27 through 29 that many points lying at the left of the Lyapunov stable regions of the gyrostat have become unstable due to the addition of the inertia imbalance $\bar{e}$ of body B_2. As $\bar{e}$ increases from 0.1 to 0.6 to 0.9 the number of unstable points within the Lyapunov stable regions also increase. But we must also note that when $\bar{e} = 0.6$ or 0.9, some of the points just within the unstable smokestack regions have become at least infinitesimally stable. Nevertheless, it is the effect on the Lyapunov stable regions that is of most importance, since these regions are known to be stable for the full non-linear system.

4.3 Some Concluding Remarks

The present Floquet analysis seems to indicate the following facts:

(a) For all practical purposes, if $0 \leq \bar{c} \leq 0.1$ and $-1 \leq \beta \leq 2$, it appears that stability-instability results for the extended gyrostat are very similar to those of a gyrostat.

(b) In the cases where $\bar{c} > 0.1$ and $\beta > 0$, we might expect drastic changes in certain areas of the stability-instability regions when inertial imbalance of B_2 (i.e. $\bar{e} \neq 0$) is introduced. The most significant of these is the effect on the Lyapunov stable region for low values of i_x which seems to contain more and more unstable points with increasing $\bar{e}$.

(c) For the case $\beta = -1.0$ the addition of inertial imbalance of B_2 seems to have no effect on the Lyapunov stable regions and seems to enhance infinitesimal stability inside the regions which were indicated to be unstable in the case of a gyrostat.

4.4 An Illustrative Example

To exemplify the effect of relative spin in conjunction with rotor inertial imbalance let us consider the following illustrative example:

Suppose we have an unsymmetrical rigid body with a fixed point which has an inertial imbalance ratio of $k = 0.7$ and a nondimensional inertia of $i_x = 0.6$, which is just inside the left hand possible rigid body limits.

If this body is placed in the vertical equilibrium position and given a spin about the vertical of $S = 2.0$, where S is the nondimensional spin speed given in Chapter II, the resulting motion is observed to be unstable. The point $(i_x, S) = (0.6, 2.0)$ clearly lies within the instability zone of Figure 7. Now if we allow a rotor to be on board in the orientation described in Chapter III and give this rotor a relative spin of $\beta = -1$, opposite in sense to that the main body, the motion of the resulting gyrostat system is observed to be unstable as shown in Figure 6. However, if the rotor is given a relative spin of $\beta = 1$ or $\beta = 2$. Figures 8 and 9 indicate that these motions are stable. This indicates that relative spin when used properly can enhance stability.

To demonstrate the effects of inertial imbalance of the rotor we now consider the present system in which we allow for an unsymmetrical rotor as described in Section 4.1. Hence if we have $(i_x, s, k) = (0.6, 2.0, 0.7)$ and introduce the nondimensional parameters $\bar{c}$ and $\bar{e}$ of this chapter, we can observe the effects of rotor inertia imbalance upon the system. Let us consider the case $\beta = 1.0$. If we allow an axial inertia ratio of $\bar{c} = 0.1$, we observe in Figure 20 that even for an inertia imbalance as high as $\bar{e} = 0.9$, there is no indication that the ensuing motion is made unstable. Nevertheless if the axial inertia ratio is increased to $\bar{c} = 0.3$, instability of the systems motion is strongly indicated in Figure 23. This example clearly illustrates that certain inertial imbalance of the rotor can be tolerated before a rotor stabilized device is made unstable due to this imbalance.

APPENDIX A

THEOREM ON ANGULAR MOMENTUM OF A GYROSTAT WITH A FIXED POINT

Theorem: Let $\mathcal{G}$ be a gyrostat consisting of coupled rigid bodies $B_1,\ldots,B_n$ in which B_1 has a point O fixed in an inertial reference frame $\mathcal{F}$. Let the orthogonal axes (x_o,y_o,z_o) be fixed in B_1 and principal for $\mathcal{G}$ with respect to O in which the z_o-axis contains the mass center G of $\mathcal{G}$. Also, let $B_2,\ldots,B_n$ be dynamically equivalent to axisymmetric bodies whose axes of symmetry are fixed in B_1 and necessarily contain their mass centers. Then

$$\vec{H}_o^{\mathcal{G}} = \vec{H}_o^{*} + \vec{h} , \tag{A.1}$$

where $\vec{H}_o^{\mathcal{G}}$ denotes the angular momentum of about O, $\vec{H}_o^{*}$ denotes the angular momentum of $\mathcal{G}$ about O as if $B_2,\ldots,B_n$ were fixed in B_1 and $\vec{h}$ denotes the vector sum of the angular momenta of $B_2,\ldots,B_n$ about their respective mass centers in their motion relative to B_1.

Proof: Let P be an arbitrary point of $\mathcal{G}$ and let dm_p be a differential element of mass containing P. Denote the mass center of body B_i by G_i $(i = 2,\ldots,n)$. Hence, we can write

$$\vec{r}_p = \vec{r}_{G_i} + \vec{\rho} \qquad i = 2,\ldots,n , \tag{A.2}$$

where $\vec{r}_p$ is the vector from O to P, $\vec{r}_{G_i}$ is the vector from O to G_i, and $\vec{\rho}_i$ is the vector from G_i to P.

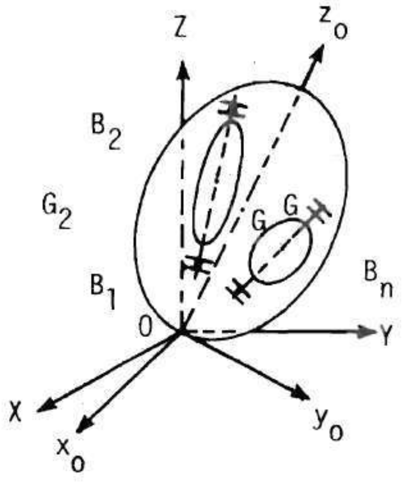

Figure 30. An Arbitrary Gyrostat with a Fixed Point.

Let ${}^{F}\vec{\omega}^{B_i}$ be the angular velocity of B_i in the inertial frame F, and let ${}^{B_1}\vec{\omega}^{B_i}$ be the angular velocity of $B_i (i = 2,\ldots,n)$ relative to B_1. Then the velocity $\vec{V}_p$ of the differential element dm_p is given by

$$\vec{V}_p = \begin{cases} {}^{F}\dfrac{d}{dt}\,\vec{r}_p \quad \text{for} \quad P \in B_1 \\ {}^{F}\dfrac{d}{dt}\,(\vec{r}_{G_i} + \vec{\rho}_i) \quad \text{for} \quad P \in B_i \ (i = 2,\ldots,n) \end{cases} \quad . \tag{A.3}$$

Since the vector $\vec{r}_p$ is constant in B_1 whenever $P \in B$, and $\vec{r}_{G_i}$ is always constant in B_1 while $\vec{\rho}_i$ is constant in B_i, we clearly obtain

$$\vec{V}_p = \begin{cases} {}^{F}\vec{\omega}^{B_1} \times \vec{r}_p \quad \text{for} \quad p \in B_1 \\ {}^{F}\vec{\omega}^{B_1} \times \vec{r}_{G_i} + {}^{F}\vec{\omega}^{B_i} \times \vec{\rho}_i \quad \text{for } P \in B_i \ (i = 2,\ldots,n) \end{cases} \quad . \tag{A.4}$$

By the additional theorem for angular velocity

$$\vec{\omega}^{F\,B_i} = \vec{\omega}^{F\,B_1} + \vec{\omega}^{F\,B_i} \qquad (i = 2,\ldots,n) \quad . \tag{A.5}$$

Hence Equation (A.4) now becomes

$$\vec{v}_p = \begin{cases} \vec{\omega}^{F\,B_1} \times \vec{r}_p & \text{for } P \in B_1 \\ \vec{\omega}^{F\,B_1} \times \vec{r}_p + \vec{\omega}^{B_1\,B_i} \times \vec{\rho}_i & P \in B_i \quad (i = 2,\ldots,n) \ , \end{cases} \tag{A.6}$$

which represents the velocity of an arbitrary point P of the gyrostat G.

By the definition of angular momentum

$$\vec{H}_o^{G} = \int_G (\vec{r}_p \times \vec{v}_p)dm_p \quad , \tag{A.7}$$

where the integration is over the entire gyrostat G.

Since $B_1,\ldots,B_n$ have no material points in common, we can write

$$\vec{H}_o^{G} = \sum_{i=1}^{n} \int_{B_i} (\vec{r}_p \times \vec{v}_p)dm_p$$

$$= \int_{B_1} (\vec{r}_p \times \vec{v}_p)dm_p + \sum_{i=2}^{n} \int_{B_i} (\vec{r}_p \times \vec{v}_p)dm_p \quad . \tag{A.8}$$

From Equation (A.6) it is clear that Equation (A.8) becomes

$$\vec{H}_o^{\,G} = \int_{B_1} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p)]dm_p$$

$$+ \sum_{i=2}^{n} \int_{B_i} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p + {}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)]dm_p \quad , \quad (A.9)$$

which obviously can be written in the form

$$\vec{H}_o^{\,G} = \sum_{i=1}^{n} \int_{B_i} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p)]dm_p$$

$$+ \sum_{i=2}^{n} \int_{B_i} [\vec{r}_p \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)]dm_p \quad . \qquad (A.10)$$

Using Equation (A.2) in Equation (A.10), we obtain

$$\vec{H}_o^{\,G} = \sum_{i=1}^{n} \int_{B_i} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p)]dm_p$$

$$+ \sum_{i=2}^{n} \int_{B_i} [(\vec{r}_{G_i} + \vec{\rho}_i) \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)]dm_p$$

$$= \sum_{i=1}^{n} \int_{B_i} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_1} \times \vec{r}_p)]dm_p$$

$$+ \sum_{i=2}^{n} [\vec{r}_{G_i} \times ({}^{B}\vec{\omega}^{B_i} \times \vec{\rho}_i)]dm_p$$

$$+ \sum_{i=2}^{n} \int_{B_i} [\vec{\rho}_i \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)] dm_p \quad . \tag{A.11}$$

Since G_i is the mass center of $B_i (i = 2,\ldots,n)$, it is clear that

$$\int_{B_i} \vec{\rho}_i \, dm_p = 0 \qquad (i = 2,\ldots,n) \quad . \tag{A.12}$$

Therefore, since $\vec{r}_{G_i}$ and ${}^{B_1}\vec{\omega}^{B_i}$ are constant vectors with regard to spatial integration over B_i,

$$\int_{B_i} [\vec{r}_{G_i} \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)] dm_p$$

$$= \vec{r}_{G_i} \times \left({}^{B_1}\vec{\omega}^{B_i} \times \int_{B_i} \rho_i \, dm_p \right) = 0 \qquad (i = 2,\ldots,n) \quad . \tag{A.13}$$

Hence, Equation (A.11) reduces to

$$\vec{H}_O^G = \sum_{i=1}^{n} \int_{B_i} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_i} \times \vec{r}_p)] dm$$

$$+ \sum_{i=2}^{n} \int_{B_i} [\vec{\rho}_i \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)] dm_p \quad . \tag{A.14}$$

Now, if we define

$$\vec{H}_O^* = \sum_{i=1}^{n} [\vec{r}_p \times ({}^{F}\vec{\omega}^{B_i} \times \vec{r}_p)] dm_p \tag{A.15}$$

and

$$\vec{h} = \sum_{i=1}^{n} \int_{B_i} [\vec{\rho}_i \times ({}^{B_1}\vec{\omega}^{B_i} \times \vec{\rho}_i)]dm_p \ ; \qquad (A.16)$$

then

$$\vec{H}_O^G = \vec{H}_O^* + \vec{h} \ , \qquad (A.17)$$

where it is clear that $\vec{H}_O^*$ is the angular momentum of G about O as if $B_2,\ldots,B_n$ were fixed in B, and that $\vec{h}$ is the sum of the angular momenta of $B_2,\ldots,B_n$ relative to B_1 about their axes which are fixed in B_1.

The above theorem is referred to in the Soviet literature as Koenig's Theorem.

APPENDIX B

AN UNSYMMETRICAL SATELLITE WITH AN UNSYMMETRICAL ROTOR IN A CIRCULAR ORBIT ABOUT THE EARTH

In Figure 31 we consider a satellite $\mathcal{S}$ composed of two rigid bodies B_1 and B_2 where G is the mass center of each of the bodies, and thus of the satellite. Also, we designate the principal axes of B_1 as (X_1, X_2, X_3), the corresponding principal moments of inertia being

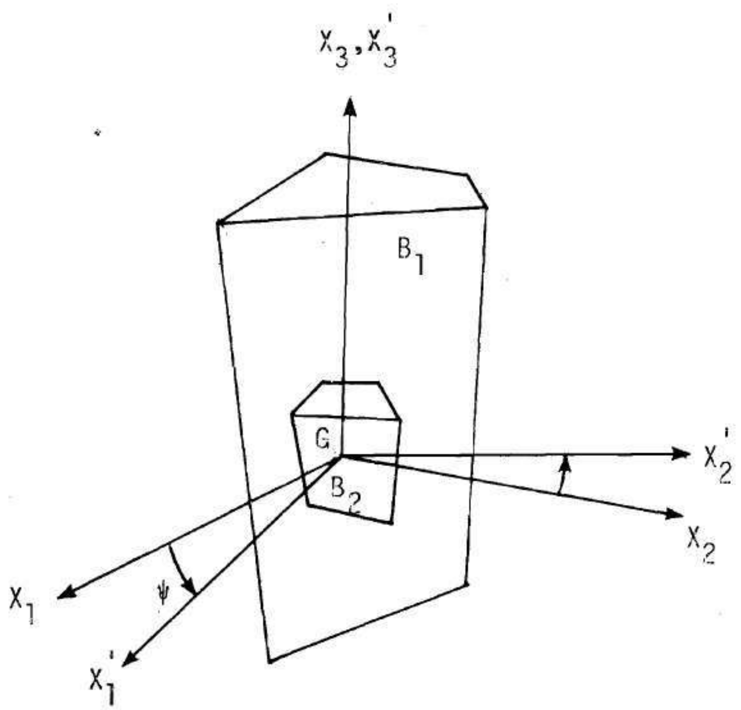

Figure 31. Schematic Representation of the Satellite.

$(I_1^{B_1}, I_2^{B_1}, I_3^{B_1})$. Furthermore, we let (x_1', x_2', x_3') represent the principal axes of B_2 with corresponding principal moments of inertia $(I_1^{B_2}, I_2^{B_2}, I_2^{B_2})$. Here X_3 and X_3' are permanently coincident, so that the only possible motions of B_2 relative to B_1 are rotations about this common axis.

Orbital reference axes (A_1, A_2, A_3) are shown in Figure 32. The

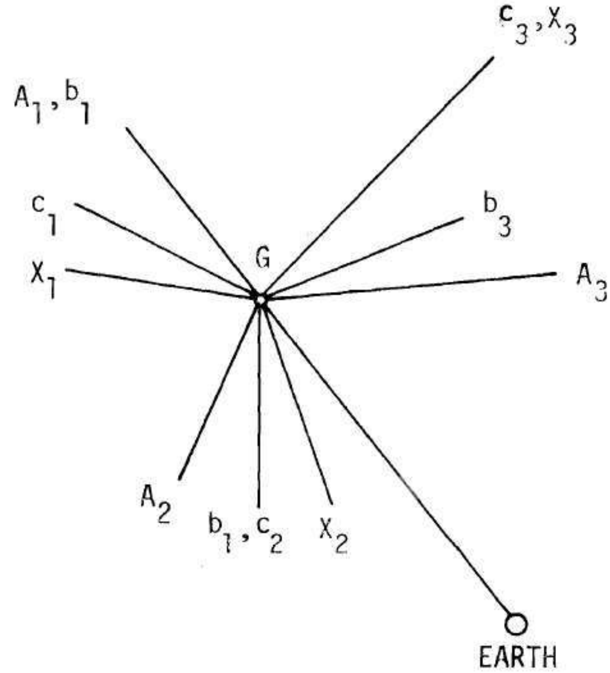

Figure 32. Reference Axes and Attitude Angles.

line passing through the center of the earth and the mass center G of the satellite is A_1; A_2 is tangent to the assumed circular orbit; and A_3 is perpendicular to the orbital plane. Any desired orientation of the satellite relative to these axes can be produced by first aligning

X_i with A_i, $i = 1,2,3$; next, performing a right-handed rotation of B_1 of amount θ_1 about A_1 bringing X_i into coincidence with b_i, $i = 1,2,3$, and following this with rotations of amount θ_2 about b_2 leading to C_1,C_2,C_3 and θ_3 about C_3 bringing b_1 into the final position.

The angle φ between A_3, the normal to the orbital plane, and the body-fixed axis X_3 depends on θ_1 and θ_2 but not on θ_3. Consequently, expressions used to study motions during which X_3 remains nearly aligned with A_3 shall be linearized in θ_1 and θ_2.

Denoting ${}^{F}\vec{\omega}^{B_1}$ and ${}^{F}\vec{\omega}^{B_2}$ as the linearized angular velocities of B_1 and B_2, respectively, in the inertial frame F, it follows from the addition theorem for angular velocities that

$$
{}^{F}\vec{\omega}^{B_2} = {}^{F}\vec{\omega}^{B_1} + {}^{B_1}\vec{\omega}^{B_2} , \tag{B.2}
$$

where ${}^{B_1}\vec{\omega}^{B_2}$ is the angular velocity of B_2 relative to B_1.

Let $(\omega_1,\omega_2,\omega_3)$ be the components of ${}^{F}\vec{\omega}^{B_1}$ referred to the axes (x_1,x_2,x_3) fixed in B_1. Thus, the components of ${}^{F}\vec{\omega}^{B_2}$ referred to the same axes obviously become $(\omega_1,\omega_2,\omega_3 + \omega)$ where $\omega = \dot{\psi}$ is the angular velocity of B_2 relative to B_1, about their common axis. In the present analysis, we are assuming $\dot{\psi} = \text{const}$. The components of ${}^{F}\vec{\omega}^{B_1}$ are given by

$$
\omega_1 = \Omega_s(\theta_1\sin\theta_3 - \theta_2\cos\theta_3) + \dot{\theta}_1\cos\theta_3 + \dot{\theta}_2\sin\theta_3 , \tag{B.3a}
$$

$$
\omega_2 = \Omega_s(\theta_1\cos\theta_3 + \theta_2\sin\theta_3) - \dot{\theta}_1\sin\theta_3 + \dot{\theta}_2\cos\theta_3 , \tag{B.3b}
$$

$$
\omega_3 = \Omega_s + \dot{\theta}_3 , \tag{B.3c}
$$

where Ω_s is the constant orbital angular speed of the satellite S. It can easily be shown that

$$\Omega_s = \sqrt{\frac{G_e\ M_e}{R^3}} = \sqrt{\frac{g\ R_e^2}{R^3}} \quad , \tag{B.4}$$

in which G_e is the gravitational constant, M_e the mass of the earth, R_e the radius of the earth, and R the orbit radius.

We should keep in mind that the angular momentum components $(\omega_1, \omega_2, \omega_3)$ of Equations (B.3a - B.3c) have been linearized in θ_1 and θ_2.

If we represent the unit vectors along the axes (x_1, x_2, x_3) as $(\hat{i}_1, \hat{i}_2, \hat{i}_3)$, then we can write the angular momentum of B_1 about G as

$$\vec{H}_G^{B_1} = I_1^{B_1}\omega_1\hat{i}_1 + I_2^{B_1}\omega_2\hat{i}_2 + I_3^{B_1}\omega_3\hat{i}_3 \quad . \tag{B.5}$$

Likewise, the components of the angular momentum of body B_2 about G referred to the same axes can be written as

$$\begin{aligned}\vec{H}_G^{B_2} &= \{[I_1^{B_1} - (I_1^{B_2} - I_2^{B_2})\sin^2 x]\omega_1 + (I_1^{B_2} - I_2^{B_2})\sin\psi\cos\psi\omega_2\}\hat{i}_1 \\ &+ \{(I_1^{B_2} - I_2^{B_2})\sin\psi\cos\psi\omega_1 + [I_1^{B_2} + (I_1^{B_2} - I_2^{B_2})\sin^2\psi]\omega_2\ \hat{i}_2 \\ &+ I_3^{B_2}(\omega_3 + \omega)\hat{i}_3 \ ,\end{aligned} \tag{B.6}$$

where

$$\psi = \omega t \ .$$

Now, if $\vec{H}_G^{S}$ denotes the total angular momentum of the satellite S about its mass center, then it follows from the definition of angular momentum that

$$\vec{H}_G^{S} = \vec{H}_G^{B_1} + \vec{H}_G^{B_2} \quad . \tag{B.7}$$

Hence, substituting Equations (B.5,B.6) into Equation (B.7) and rearranging terms, we clearly obtain

$$\begin{aligned}\vec{H}_G^{S} = \{&[I_1^{B_1} + I_1^{B_2} - (I_1^{B_2} - I_2^{B_2})\sin^2\psi]\omega_1 \\ &+ (I_1^{B_2} - I_2^{B_2})\sin\psi\cos\psi\,\omega_2\}\hat{i}_1 \\ &+ \{(I_1^{B_2} - I_1^{B_2})\sin\psi\cos\psi\,\omega_1 \\ &+ [I_2^{B_1} + I_2^{B_2} + (I_1^{B_2} - I_2^{B_2})\sin^2\psi]\omega_2\}\ \hat{i}_2 \\ &+ [(I_3^{B_1} + I_3^{B_2})\omega_3 + I_3^{B_2}\omega\}\ \hat{i}_3 \ ,\end{aligned} \tag{B.8}$$

which represents the total angular momentum of the satellite S about its mass center G.

For the purpose of simplification let us introduce the following notation:

$$I_i = I_i^{B_1} + I_i^{B_2} \qquad i = 1,2,3$$

$$e_s = I_1^{B_2} - I_2^{B_2} . \tag{B.9}$$

Therefore, we can rewrite Equation (B.8) in the form

$$\vec{H}_G^S = [(I_1 - e_s \sin^2\psi)\omega_1 + e_s \sin\psi\cos\psi\omega_2]\hat{i}_1$$

$$+ [e_s \sin\psi\cos\psi\omega_1 + (I_2 + e_s \sin^2\psi)\omega_2]\hat{i}_2 + (I_3\omega_3 + I_3^{B_2}\omega)\hat{i}_3 . \tag{B.10}$$

We are assuming here that the only external torques acting on the satellite are those due to the earth's gravitational field. If a body in a central force field does not possess spherical symmetry, differential-gravity torques are present which is the case here. In the literature on spacecraft dynamics, these torques are referred to as "gravity-gradient torques" or sometimes just "gravity torques."

Let $\vec{M}_G^{B_1}$ represent the moment exerted on body B_1 about G due only to the gravitational torque. Referred to the (x_1, x_2, x_3) axes the components $({}_1M_G^{B_1}, {}_2M_G^{B_1}, {}_3M_G^{B_1})$ take the form

$${}_1M_G^{B_1} = 3\Omega_s^2(I_2^{B_1} - I_3^{B_1})\theta_2\sin\theta_3 \quad , \tag{B.11a}$$

$${}_2M_G^{B_1} = 3\Omega_s^2(I_1^{B_1} - I_3^{B_1})\theta_2\cos\theta_3 \quad , \tag{B.11b}$$

$${}_3M_G^{B_1} = 3\Omega_s^2(I_1^{B_1} - I_2^{B_1})\sin\theta_3\cos\theta_3 \ , \tag{B.11c}$$

where ${}_iM_G^{\theta_1}$, $i = 1,2,3$, have been linearized in θ_1 and θ_2.

Similarly, if $\vec{M}_G^{B_2}$ denotes the gravitational torque exerted on body

B_2 about G its components $({}_1M_G^{B_2}, {}_2M_G^{B_2}, {}_3M_G^{B_2})$, referred to axes (x_1, x_2, x_3) which are fixed in body B_1, are found to be more complex. These components are as follows:

$$ {}_1M_G^{B_2} = 3\Omega_s^2[-I_s^{B_2}\sin\theta_3 + (I_2^{B_2}\cos^2\psi + I_1^{B_2}\sin^2\psi)\sin\theta_3 $$

$$ + (I_2^{B_2} - I_1^{B_1})\cos\theta_3\sin\psi\cos\psi]\theta_2 \ , \qquad \text{(B.12a)} $$

$$ {}_2M_G^{B_2} = 3\Omega_s^2[-I_3^{B_2}\cos\theta_3 + (I_2^{B_2}\sin^2\psi + I_1^{B_2}\cos^2\psi)\cos\theta_3 $$

$$ + (I_2^{B_2} - I_2^{B_2})\sin\theta_3\sin\psi\cos\psi]\theta_2 \ , \qquad \text{(B.12b)} $$

$$ {}_3M_G^{B_2} = \frac{3}{2}\Omega_s^2(I_1^{B_2} - I_2^{B_2})(\sin 2\theta_3\cos 2\psi + \sin 2\psi\cos 2\theta_3) \ , \qquad \text{(B.12c)} $$

where ${}_iM_G^{B_2}$, $i = 1,2,3$, have also been linearized in θ_1 and θ_2.

The external torque exerted on body B_1 by body B_2 is counteracted by the external torque exerted on body B_2 by body B_1, i.e., these two torques become internal for the total system. Thus, if we let $\vec{M}_G$ represent the total external moment exerted on the satellite S about its mass center G, then it becomes clear that

$$ \vec{M}_G^S = \vec{M}_G^{B_1} + \vec{M}_G^{B_2} \ . \qquad \text{(B.13)} $$

From Equations (B.11-B.12), we can write the components $({}_1M_G, {}_2M_G, {}_3M_G)$ of $\vec{M}_G^S$ as follows:

$$ {}_1M_G = {}_1M_G^{B_1} + {}_1M_G^{B_2} = 3\Omega_s^2[(I_2^{B_1} - I_3^{B_1} - I_3^{B_2})\sin\theta_3 $$

$$ + (I_2^{B_2}\cos^2\psi + I_1^{B_2}\sin^2\psi)\sin\theta_3 $$

$$ + (I_2^{B_2} - I_1^{B_2})\sin\psi\cos\psi\cos\theta_3]\theta_2 \quad , \qquad \text{(B.14a)} $$

$$ {}_2M_G = {}_2M_G^{B_1} + {}_2M_G^{B_2} = 3\Omega_s^2[(I_1^{B_1} - I_3^{B_1} - I_3^{B_2})\cos\theta_3 $$

$$ + (I_2^{B_2}\sin^2\psi + I_1^{B_2}\cos^2\psi)\cos\theta_3 $$

$$ + (I_2^{B_2} - I_1^{B_2})\sin\psi\cos\psi\sin\theta_3]\theta_2 \quad , \qquad \text{(B.14b)} $$

$$ {}_3M_G = {}_3M_G^{B_1} + {}_3M_G^{B_2} = 3\Omega_s^2[(I_1^{B_1} - I_2^{B_1})\sin2\theta_3 $$

$$ + (I_1^{B_2} - I_2^{B_2})(\sin2\theta_3\cos2\psi + \sin2\psi\cos2\theta_3)]. \qquad \text{(B.14c)} $$

Introducing the notation of (B.9), Equations (B.14) become

$$ {}_1M_G = 3\Omega_s^2[(I_2 - I_3)\sin\theta_3 - e_s\cos(\theta_3 + \psi)\sin\psi]\theta_2 \;, \qquad \text{(B.15a)} $$

$$ {}_2M_G = 3\Omega_s^2[(I_1 - I_3)\cos\theta_3 - e_s\sin(\theta_3 + \psi)\sin\psi]\theta_2 \;, \qquad \text{(B.15b)} $$

$$ {}_3M_G = 3\Omega_s^2[(I_1 - I_2 - e_s)\sin2\theta_3 + e_s\cos2(\theta_3 + \psi)]\theta_2 \;. \qquad \text{(B.15c)} $$

The moment equation for the entire satellite S with respect to its mass center G retains the form

$$ {}^{F}\dot{\vec{H}}_G^S = {}^{B_1}\dot{\vec{H}}_G^S + {}^{F}\vec{\omega}^{B_1} \times \vec{H}_G^S = \vec{M}_G^S \quad , \tag{B.16} $$

in which ${}^{F}\dot{\vec{H}}_G$ and ${}^{B_1}\dot{\vec{H}}_G$ indicate differentiation of the total angular momentum of the satellite with respect to time in the F and B_1 frames respectively.

Substituting the expressions for $\vec{H}_G^S$, ${}^{F}\vec{\omega}^{B_1}$, and $\vec{M}_G^S$ into Equation (B.16) and performing the given operations yields, after some algebraic manipulations, the following differential equations:

$$
\begin{aligned}
&(I_1 - e_s\sin^2\psi)\dot{\omega}_1 + \frac{1}{2}e_s\sin 2\psi\dot{\omega}_2 \\
&\quad = \left[\omega + \frac{1}{2}(\Omega_s + \dot{\theta}_3)\right]e_2\sin 2\psi\omega_1 \\
&\qquad - [(I_3^{B_2} + e_s\cos 2\psi)\omega + (I_3 - I_2 - e_s\sin^2\psi)(\Omega_s + \dot{\theta}_3)]\omega_2 \\
&\qquad + 3\Omega_s^2[(I_2 - I_3)\sin\theta_3 - e_s\cos(\theta_3 + \psi)\sin\psi]\theta_2 \quad ,
\end{aligned} \tag{B.17}
$$

$$
\begin{aligned}
&\frac{1}{2}e_s\sin 2\psi\dot{\omega}_1 + (I_2 + e_s\sin^2\psi)\dot{\omega}_2 \\
&\quad = [(I_3^{B_2} - e_s\cos 2\psi)\omega - (I_1 - I_3 - e_s\sin^2\psi)(\Omega_s + \dot{\theta}_3)]\omega_1 \\
&\qquad - \left[\omega + \frac{1}{2}(\Omega_s + \dot{\theta}_3)\right]e_s\sin 2\psi\ \omega_2 \\
&\qquad + 3\Omega_s^2[(I_1 - I_3)\cos\theta_3 - e_s\sin(\theta_3 + \psi)\sin\psi]\theta_2 \quad ,
\end{aligned} \tag{B.18}
$$

$$I_3\ \ddot{\theta}_3 - \frac{3}{2}\ \Omega_3^2(I_1 - I_2 - e_s)\sin 2\theta_3 - \frac{3}{2}\ \Omega_3^2\ e_s \sin\ 2(\theta_3 + \psi) = 0\ . \tag{B.19}$$

Now, we represent the determinant of the coefficients of $\dot{\omega}_1$ and $\dot{\omega}_2$ of Equations (B.17,B.18) by Δ_s where

$$\Delta_s = I_1 I_2 + (I_1 - I_2 - e_s)e_s \sin^2\psi\ . \tag{B.20}$$

It can easily be shown that Δ_s is always positive.

Therefore, solving for $\dot{\omega}_1$ and $\dot{\omega}_2$ from Equations (B.17, B.18) by using Cramer's rule, we obtain the following linear differential equations with variable coefficients:

$$\begin{aligned}
\dot{\omega}_1 = {} & \frac{1}{\Delta_s}\left[\left(I_2 - \frac{1}{2}\ I_3^{B_2 2} + \frac{1}{2}\ e_s\right)\omega + \frac{1}{2}\ (I_1 + I_2 - I_3)(\Omega_s + \dot{\theta}_3)\right] e_s \sin 2\psi\omega_1 \\
& + \frac{1}{\Delta_s}\ \{(-\ I_3 I_3^{B_2 2} - I_2 e_s \cos 2\psi - I_3^{B_2 2} e_s \sin^2\psi + e_s^2 \sin^2\psi)\omega \\
& + [-\ I_2(I_3 - I_2) - (I_3 - 2I_2)e_s \sin^2\psi + e_s^2 \sin^2\psi](\Omega_s + \dot{\theta}_s)\}\omega_2 \\
& + \frac{1}{\Delta_s} \cdot 3\Omega_s^2\ \{(I_2 + e_s \sin^2\psi)[(I_2 - I_3)\sin\theta_3 - e_s \cos(\theta_s + \psi)\sin\psi] \\
& - \frac{1}{2}\ e_s \sin 2\psi[(I_1 - I_3)\cos\theta_3 - e_s \sin(\theta_3 + \psi)\sin\psi]\}\theta_2\ \ ,
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
\dot{\omega}_2 = {} & \frac{1}{\Delta_s}\ \{[I_1 I_3^{B_2 2} - I_3^{B_2 2}\ e_s \sin^2\psi - I_1 e_s \cos 2\psi - e_s^2 \sin^2\psi]\omega \\
& - [I_1(I_1 - I_3) - (2I_1 - I_3)e_s \sin^2\psi + e_s^2 \sin^2\psi](\Omega_s + \dot{\theta}_3)\}\omega_1 \\
& + \frac{1}{\Delta_s}\left[\left(-\ I_1 + \frac{1}{2}\ I_2^{B_2 2} + \frac{1}{2}\ e_s\right)\omega + \frac{1}{2}(I_3 - I_1 - I_2)(\Omega_s + \dot{\theta}_3)\right] e_s \sin 2\psi\omega_2
\end{aligned}$$

$$+ \frac{1}{\Delta_s} \cdot 3\Omega_s^2\{(I_1 - e_s\sin^2\psi)[(I_1 - I_3)\cos\theta_3 - e_s\sin(\theta_3 + \psi)\sin\psi]$$

$$- \frac{1}{2} e_s\sin2\psi[(I_2 - I_3)\sin\theta_3 - e_s\cos(\theta_2 + \psi)\sin\psi]\Big\}\theta_2 \quad . \tag{B.22}$$

It is now advantageous to introduce the following nondimensional parameters:

$$i_x = I_1/I_3 \ , \quad k = I_2/I_1 \ , \quad c = I_3^{B_2}/I_3 \ , \quad \bar{e}_s = e_s/I_3^{B_2} \ . \tag{B.23}$$

Thus, dividing both the numerator and denominator of Equations (B.21,B.22) by $(I_3)^2$ and making use of (B.23) we obtain

$$\dot{\omega}_1 = \frac{1}{\Delta_s^*} \Big\{\Big(ki_x - \frac{1}{2} c + \frac{1}{2} c\ \bar{e}_s\Big)\omega + \frac{1}{2} [i_x(1 + k) - 1](\Omega_s + \dot{\theta}_s)\Big\}c\ \bar{e}_s\sin2\psi\omega_1$$

$$+ \frac{1}{\Delta_s^*} \{(- ki_xc - ki_xc\bar{e}_s\cos2\psi - c^2\bar{e}_s\sin^2\psi - c^2\bar{e}_s^2\sin^2\psi)\omega$$

$$+ [-ki_x(1 - ki_x) - (1 - 2ki_x)c\ \bar{e}_s\sin^2\psi + c^2\bar{e}_s^2\sin^2\psi](\Omega_s + \dot{\theta}_3)\}\omega_2$$

$$+ \frac{1}{\Delta_s} 3\Omega_s^2\{ki_x + c\bar{e}_s\sin^2\psi)[(ki_x - 1)\sin\theta_3 - c\ \bar{e}_s\cos(\theta_3 + \psi)\sin\psi]$$

$$- \frac{1}{2} c\ \bar{e}_s\sin2\psi[(i_x - 1)\cos\theta_3 - c\ \bar{e}_s\sin(\theta_3 + \psi)\sin\psi]\}\theta_2 \ , \tag{B.24}$$

$$\dot{\omega}_2 = \frac{1}{\Delta_s^*} \Big\{[i_xc - c^2\bar{e}_s\sin^2\psi - i_xc\ \bar{e}_s\cos2\psi + c^2\bar{e}_s^2\sin^2\psi]\omega$$

$$- [i_x(i_x - 1) - (2i_x - 1)c\ \bar{e}\ \sin^2\psi + c^2\bar{e}^2\sin^2\psi](\Omega_s + \dot{\theta}_3)\}\omega_1$$

$$+ \frac{1}{\Delta_s^*} \Big\{\Big(- i_x + \frac{1}{2} c + \frac{1}{2} c\ \bar{e}_s\Big)\omega + \frac{1}{2} [1 - i_x(1 + k)](\Omega_s + \dot{\theta}_3)\Big\}c\ \bar{e}_s\sin2\psi\omega_2$$

$$+ \frac{3\Omega_3^2}{\Delta_s^*} \{(i_x - c\,\bar{e}\,\sin^2\psi)[(i_x - 1)\cos\theta_3 - c\,\bar{e}_s\sin(\theta_3 + \psi)\sin\psi]$$

$$- \frac{1}{2}\,c\,\bar{e}_s\sin2\psi[(ki_x - 1)\sin\theta_3 - c\,\bar{e}_s\cos(\theta_3 + \psi)\sin\psi]\}\theta_2 \; , \qquad \text{(B.25)}$$

where

$$\Delta_s^* = \Delta_s/(I_3)^2 \quad . \qquad \text{(B.26)}$$

Returning to Equations (B.3a,B.3b) and solving for $\dot{\theta}_1$ and $\dot{\theta}_2$ we obtain

$$\dot{\theta}_1 = \cos\theta_3\omega_1 - \sin\theta_3\omega_2 + \Omega_s\theta_2 \; , \qquad \text{(B.27)}$$

$$\dot{\theta}_2 = \sin\theta_3\omega_1 + \cos\theta_3\omega_2 - \Omega_s\theta_1 \; . \qquad \text{(B.28)}$$

Now let us introduce a nondimensional time parameter $\bar{\tau}$ defined by

$$\bar{\tau} = \Omega_s t \quad , \qquad \text{(B.29)}$$

where we are assuming that $\Omega_s \neq 0$. Thus, it becomes clear that

$$\frac{d}{dt} = \Omega_s \frac{d}{d\bar{\tau}} \; . \qquad \text{(B.30)}$$

Furthermore, it is also beneficial to introduce nondimensional angular velocities $\bar{\omega}_1$ and $\bar{\omega}_2$ such that

$$\omega_1 = \frac{\omega_1}{\Omega_s} \quad \text{and} \quad \bar{\omega}_2 = \frac{\omega_2}{\Omega_s} \; . \qquad \text{(B.31)}$$

Thus, if we denote differentiation with respect to the nondimensional time $\bar{\tau}$ by a prime, i.e. let $\frac{d}{d\bar{\tau}} = (\quad)'$ and make use of Equations (B.31) then Equations (B.24, B.25, B.27, B.28) can be mathematically manipulated into the following nondimensional form:

$$\bar{\omega}_1' = \frac{1}{\Delta_s^*}\left\{\left(ki_x - \frac{1}{2}c + \frac{1}{2}c\bar{e}_s\right)\frac{\omega}{\Omega_s} + \frac{1}{2}[i_x(1+k) - 1](1+\theta_3')\right\}c\,\bar{e}_s\sin 2\psi\,\bar{\omega}_1$$

$$+ \frac{1}{\Delta_s^*}\{(-ki_x c - ki_x c\,\bar{e}_s\cos 2\psi - c^2\bar{e}_s\sin^2\psi + c^2\bar{e}_s^2\sin^2\psi)\frac{\omega}{\Omega_s}$$

$$+ [-ki_x(1-ki_x) - (1-2ki_x)c\,\bar{e}_s\sin^2\psi + c^2\bar{e}_s^2\sin^2\psi](1+\theta_3')\}\bar{\omega}_2$$

$$+ \frac{3}{\Delta_s^*}\{(ki_x + c\bar{e}_s\sin^2\psi)[(ki_x - 1)\sin\theta_3 - c\,\bar{e}_s\cos(\theta_3+\psi)\sin\psi]$$

$$- \frac{1}{2}c\,\bar{e}_s\sin 2\psi[(i_x - 1)\cos\theta_3 - c\,\bar{e}_s\sin(\theta_3+\psi)\sin\psi]\}\theta_2\ , \qquad \text{(B.32)}$$

$$\bar{\omega}_2' = \frac{1}{\Delta_s^*}\left\{[i_x c - c^2\bar{e}_s\sin^2\psi - i_x c\,\bar{e}_s\cos\psi + c^2\bar{e}_s^2\sin^2\psi]\frac{\omega}{\Omega_s}\right.$$

$$\left.- [i_x(i_x - 1) - (2i_x - 1)c\,\bar{e}_s\sin^2\psi + c^2\bar{e}_s^2\sin^2\psi](1+\theta_3')\right\}\bar{\omega}_1$$

$$+ \frac{1}{\Delta_s^*}\left\{\left(-i_x + \frac{1}{2}c + \frac{1}{2}c\,\bar{e}_s\right)\frac{\omega}{\Omega_s}\right.$$

$$\left.+ \frac{1}{2}[1 - i_x(1+k)](1+\theta_3')\right\}c\,\bar{e}_s\sin 2\psi\,\bar{\omega}_2$$

$$+ \frac{3}{\Delta_s^*}\left\{(i_x - c\,\bar{e}_s\sin^2\psi)[(i_x - 1)\cos\theta_3 - c\,\bar{e}_s\sin(\theta_3+\psi)\sin\psi]\right.$$

$$\left.- \frac{1}{2}c\,\bar{e}_s\sin 2\psi[(ki_x - 1)\sin\theta_3 - c\,\bar{e}_s\cos(\theta_3+\psi)\sin\psi]\right\}\theta_2\ , \qquad \text{(B.33)}$$

$$\theta_1' = \cos\theta_3\bar{\omega}_1 - \sin\theta_3\bar{\omega}_2 + \theta_2 \quad , \tag{B.34}$$

$$\theta_2' = \sin\theta_3\bar{\omega}_1 + \cos\theta_3\bar{\omega}_2 - \theta_1 \quad . \tag{B.35}$$

Also, we can rewrite Equation (B.19) into the nondimensional form

$$\theta_3'' - \frac{3}{2}\,[i_x(1 - k) - c\,\bar{e}_s]\sin 2\theta_s - \frac{3}{2}\,c\,\bar{e}_s\sin 2(\theta_3 + \psi) = 0. \tag{B.36}$$

In Equations (B.32-B.36), we must realize that

$$\psi = \omega t = \frac{\omega}{\Omega_s}\,\bar{\tau} \quad . \tag{B.37}$$

Equations (B.32-B.36), represent the differential equations of motion for the present problem, in which we have linearized in terms of θ_1 and θ_2. These differential equations are first order and linear in the variables θ_1, θ_2, $\bar{\omega}_1$ and $\bar{\omega}_2$ and have variable coefficients.

Note that in the special case in which B_2 is symmetric about its spin axis, $\bar{e}_s = 0$ and Equations (B.32, B.33, B.36) reduce to

$$\bar{\omega}_1' = -\frac{1}{i_x}\left[c\,\frac{\omega}{\Omega_s} + (1 - ki_x)(1 + \theta_3')\right]\bar{\omega}_2 - \frac{3}{i_x}\,[(1 - ki_x)\sin\theta_3]\theta_2, \tag{B.38}$$

$$\bar{\omega}_2' = \frac{1}{ki_x}\left[c\,\frac{\omega}{\Omega_s} + (1 - i_x)(1 + \theta_s')\right]\bar{\omega}_1 - \frac{3}{ki_x}\,[(1 - i_x)\cos\theta_3]\theta_2 \quad , \tag{B.39}$$

and

$$\theta_3'' - \frac{3}{2}\,i_x(1 - k)\sin 2\theta_3 = 0 \quad . \tag{B.40}$$

Since Equations (B.34, B.35) do not contain the nondimensional parameter $\bar{e}_s$, they remain unchanged when $\bar{e}_s$ is set to equal zero.

This particular case was investigated by Kane and Mingori [40], in which they easily recognized Equation (B.40) as the differential equation of motion of a pendulum. Here, it is evident that θ_3 can be either an oscillatory or monotone function of $\bar{\tau}$ depending on the size of $|\theta_3'|$ when θ_3 is equal to zero.

It is easy to show that Equation (B.36) possesses the first integral

$$(\theta_3')^2 - \frac{3}{2}\, i_x(1 - k)\sin^2\theta_3 = \text{const.}, \tag{B.41}$$

from which it is clear that $\theta_3'(\bar{\tau})$ is a periodic function of $\bar{\tau}$ with the same period as $\sin^2[\theta_3(\bar{\tau})]$. Thus, it follows that Equations (B.30), (B.31), (B.34), and (B.35) are linear differential equations with periodic coefficients in both the "oscillatory" and "rotational" cases.

Equation (B.40) yields an elliptical integral from which the period $\bar{\tau}_o$ for the "oscillatory" case, or the quasi-period $\bar{\tau}_r$ for the "rotational" case, can be obtained. Here $\bar{\tau}_r$ is defined as the change in $\bar{\tau}$ corresponding to a change in 2π radians in θ_3.

Clearly, Equations (B.34, B.35, B.38, B.39) can be put into matrix form

$$\{X\}' = [A(\bar{\tau})]\{X\}\ , \tag{B.42}$$

where $\{X\} = \text{col.}\{\theta_1, \theta_2, \bar{\omega}_1, \bar{\omega}_2\}$ and $[A(\bar{\tau})]$ is a 4×4 matrix whose elements are continuous periodic functions $a_{ij}(\bar{\tau})$ of period $\bar{\tau}_o$ or $\bar{\tau}_r$, depending upon the case under consideration.

The matrix Equation (B.42) meets all of the requirements for the application of Floquet's theory which was used in [40] to study the stability of the system defined by Equations (B.40, B.42).

In accordance with Floquet's theory, the boundness of the solutions of Equation (B.42) depends upon the value of $\bar{\tau}_o$ (or $\bar{\tau}_r$) of the 4×4 matrix defined by the differential equation

$$[H(\bar{\tau})]' = [A(\bar{\tau})][H(\bar{\tau})] \tag{B.43}$$

and the initial conditions

$$[H(0)] = [I] , \tag{B.44}$$

where $[I]$ denotes the 4×4 unit matrix. All solutions of Equation (B.42) are bounded as $\bar{\tau} \to \infty$ if and only if the modulus of each of the four characteristic values of $[H(\bar{\tau}_o)]$(or $[H(\bar{\tau}_r)]$) is less than or equal to unity, and if, for any characteristic value λ_i such that $|\lambda_i| = 1$ the multiplicity of λ_i is equal to the nullity of the matrix $[H(\bar{\tau}_o)] - \lambda[I]$ (or $[H(\bar{\tau}_r)] - \lambda[I]$).

Consequently, given values of the parameters i_x, k, c and $\frac{\omega}{\Omega_s}$ we can determine $\bar{\tau}_o$ or $\bar{\tau}_r$ and then simultaneously perform numerical integration of Equation (B.40) and Equation (B.42), using the initial values $\theta_3(0) = 0$, $\theta_3' =$ some arbitrary value, and $[H(0)] = [I]$, and terminating the integration at $\bar{\tau} = \bar{\tau}_o$(or $\bar{\tau}_r$). Next, we can find the roots $\lambda_{1,2,3}$ of the resulting characteristic equation:

$$\det ([H(\tau_o)] - \lambda[I]) = 0 \quad ,$$

or (B.45)

$$\det ([H(\tau_r)] - \lambda[I]) = 0 \quad .$$

Once the modulus of each distinct root has been determined, then the instability decision follows.

This was the procedure followed by Kane and Mingori [40], in which they investigated both oscillatory and rotational cases. Their results showed that the inclusion of a symmetrical rotor in an unsymmetrical satellite can produce both beneficial and harmful effects regarding the attitude stability of the satellite.

More recently, this same problem was investigated by Da Silva [20], in which he was able to employ the direct method of Lyapunov to yield sufficed conditions for stability for the full nonlinear system.

However, in the case where $\bar{e}_s \neq 0$, an approach to a stability analysis is not obvious. Equations (B.32-B.35) can easily be put into the matrix form

$$\{x\}' = [B(\bar{\tau})]\{x\} , \tag{B.46}$$

where again

$$\{x\} = \text{col.}\ (\theta_1, \theta_2, \theta_3, \theta_4) .$$

If Floquet theory is to be used to obtain stability results for the system described by Equation (B.46), we must first show that $[B(\bar{\tau})]$ is a periodic matrix and that all of its components $b_{ij}(\bar{\tau})$ are continuous for all $\bar{\tau}$. However, whether or not $[B(\bar{\tau})]$ is periodic depends upon the periodicity of θ_3' and $\sin\theta_3$, which follows from the behavior of the solutions of Equation (B.32). Unfortunately, due to a lack of knowledge of the behavior of solutions for equations of the form Equation (B.32), the periodicity of θ_3' and $\sin\theta_3$ remains a question for further study. Clearly, the period of $\sin^2\psi\left(= \sin^2 \frac{\omega}{\Omega}\tau\right)$ is $T_s = \frac{\Omega_s}{\omega}\pi$. Hence, if it were possible to obtain the periods T_1 and T_2 of θ_3' and $\sin\theta_3$ respectively, then it would remain to be shown that both $\frac{T_s}{T_1}$ and $\frac{T_s}{T_2}$ are rational numbers. This condition is needed to make the entire matrix periodic. Obviously, it is unwise to depend upon a numerical scheme to obtain the periods of T_1 and T_2.

At this point, one might suggest that the Equations (B.32-B.36) be linearized in θ_3 in order to obtain stability results. Comparatively, if this is done the situation is not much improved. Equation (B.36) reduces to

$$\theta_3'' - 3[i_x(1 - k) - (1 - c\,\bar{e}_s)\cos 2\psi]\theta_3 = \frac{3}{2}\sin\psi\ , \qquad (B.47)$$

where $\psi = \frac{\omega}{\Omega_s}\bar{\tau}$. This equation is recognized as a nonhomogeneous Mathieu equation for which there exists little information in the literature regarding the periodicity of its solutions. Even if the periodicity of θ_3 could be proved and its period T_o could be obtained, again we would have to show that the ratio of T_o to $\frac{\Omega_s}{\omega}\pi$ is a rational number, which eliminates any numerical computation of T_o.

In a paper by Kane [44], it was shown that a linearization of θ_3 for the problem of the unsymmetric satellite (without a rotor) in circular orbit led to incorrect stability results. This suggests that it would be unwise to repeat such a linearization for the present, more involved, problem.

Therefore, due to the numerous mathematical difficulties that have arisen, the stability analysis for the present satellite problem was abandoned at this point. This appendix is included so that the governing differential equations might be of help to future workers on this problem.

APPENDIX C

THE DIRECT METHOD OF LYAPUNOV[1]

C.1 Discussion of the Method

In many cases, investigators utilize the variational approach to answer the question of the stability of a dynamical system. An alternate approach to such an investigation is provided by the direct method of Lyapunov, also known as Lyapunov's second method. While neither approach is restricted to linear systems or requires knowledge of the solution of the differential equations, the direct method of Lyapunov is much stronger in the sense that it does not restrict the investigation to a small neighborhood of the equilibrium as is required by the variational approach. Unfortunately, the direct method of Lyapunov has a major drawback in that it necessitates the construction of a testing function, known as a "Lyapunov function" or "Lyapunov test function", which may not always be possible. This nonuniqueness of the Lyapunov function leads to a large degree of flexibility in its selection. Presently, there exists no general procedure for constructing such functions except for linear autonomous systems and linear non-autonomous second-order systems (see Reference [45]). Nevertheless, the direct method of Lyapunov represents an important tool in the analysis of stability problems.

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[1]The theorems and definitions which follow in this appendix are taken from [7].

C.2 Stability Concepts

Let us consider an autonomous dynamical system which can be written in vector form

$$\dot{\vec{x}} = \vec{X}(\vec{x}) \quad , \tag{C.1}$$

where $\vec{x}$ and $\vec{X}$ are real continuous n-vectors and $\vec{X}(\vec{0}) = \vec{0}$. We define the norm or length of $\vec{x}$ by

$$\| \vec{x} \| = \sqrt{x_1^2 + \cdots + x_n^2} \quad , \tag{C.2}$$

in which the x_i are the components of $\vec{x}$. The vector $\vec{X}$ is assumed to satisfy a Lipschitz condition in a spherical domain $D_h : \| \vec{x} \| < h$ with center at the origin of the Euclidean n-space, where h is a positive constant. This condition ensures uniqueness of solutions within D_h.

The vector $\vec{x} = \vec{a}$ with the property that $\vec{X}(\vec{a}) = \vec{0}$ for all $\epsilon \geq 0$, is called an "equilibrium point" or "equilibrium solution" of the dynamical system. It is clear that by a simple linear transformation, any equilibrium point can be made to coincide with the origin.

We say that the origin is a stable equilibrium point of (C.1) if for every $\epsilon > 0$ and arbitrary time t_o there exists a $\delta > 0$ such that if $\| \vec{x}(t_o) \| < \delta$, then $\| \vec{x}(t) \| < \epsilon$ for all $t \geq t_o$.

A simple geometric interpretation of this definition is illustrated in Figure 33 in which we take the origin as an equilibrium point for the autonomous dynamical system described by Equation (C.1). Here, we say that the origin is a "stable" point if for every sphere $S_R : \|\vec{x}\| < R$ we can find a second sphere S_r: $\|\vec{x}\| < r$ such that if any path initiates

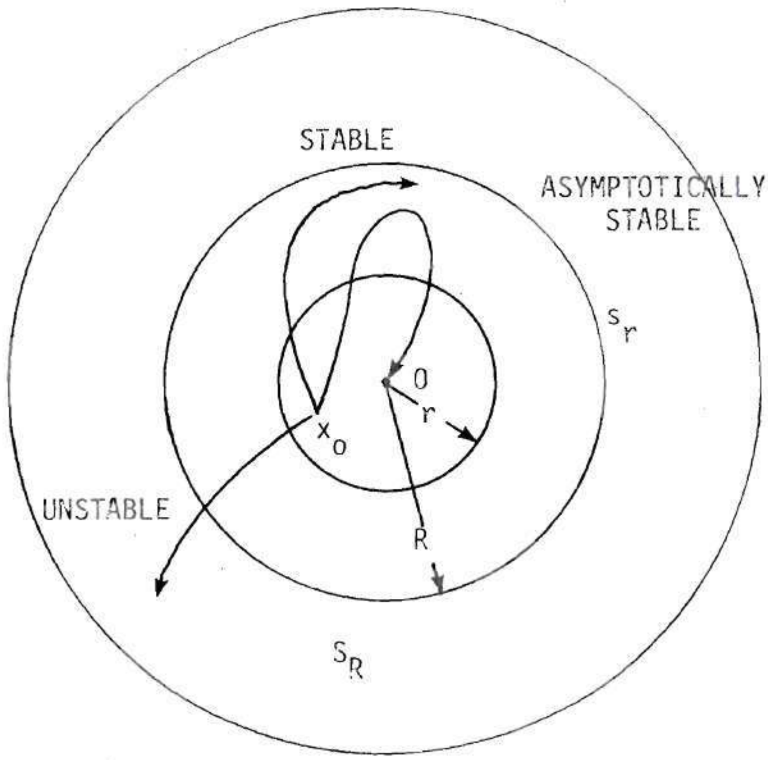

Figure 33. Geometric Interpretation of Stability.

at a point $\vec{x}_o = \vec{x}(t_o)$ inside S_r it remains in the sphere S_R for all t thereafter. Otherwise, we say the origin is unstable.

If, in addition to being stable, every path that begins within the sphere tends to the origin as time increases, then we say that the origin is asymptotically stable.

It is noteworthy to mention that when the variational method is used, the allowable perturbations are restricted to a small neighborhood about the equilibrium point, such that the equations of motion represent a linear approximation. In such a case, stability must be referred to as "infinitesimal stability".

Let $V(\vec{x})$ be a real continuous function defined in a spherical domain D_h: $\| \vec{x} \| < h$ having the properties:

(a) $V(\vec{0}) = 0$,

(b) $V(\vec{x})$ possesses continuous first partial derivatives with respect to all its variables in D_h.

Before introducing two important theorems used frequently in a Lyapunov stability analysis, it is helpful to note the following definitions:

Definition C.1. The function $V(\vec{x})$ is called positive (negative) definite in a domain D_h if $V(\vec{x}) > 0$ (< 0) for all $\vec{x} \neq 0$ and $V(\vec{0}) = \vec{0}$.

Definition C.2. The function $V(\vec{x})$ is called positive (negative) semi-definite in a domain D_h if $V(\vec{x}) \geq 0$ (≤ 0) for all $\vec{x}$ in D_h.

Definition C.3. The function $V(\vec{x})$ is called indefinite in a domain D_h if it can assume both positive and negative values in D_h.

Definition C.4. The total derivative of the function $V(\vec{x})$ with respect to time is defined to be

$$\dot{V}(\vec{x}) = \frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} X_i = \nabla V \cdot \vec{X} . \qquad (C.3)$$

In reference to these definitions, we now introduce two Lyapunov stability theorems.

Theorem 1[1]. If there exists for the system (C.1) a positive (negative) definite function $V(\vec{x})$ such that along every trajectory of system (C.1) $\dot{V}(\vec{x})$ is negative (positive) semidefinite, then the origin is a stable point.

Proof: First, we assume that there exist a function $V(\vec{x})$ which is positive definite in a spherical domain D_h: $\|\vec{x}\| < h$ and such that its total time derivative $\dot{V}(\vec{x})$ is negative semidefinite in D_h.

Now let us choose an arbitrary small positive number ϵ such that $\epsilon < h$ and $\|\vec{x}\| = \epsilon$ denotes a sphere centered at the origin and of radius ϵ. If we set $V_\epsilon = \inf V(\vec{x})$ on the sphere $\|\vec{x}\| = \epsilon$, it follows that

$$V(\vec{x}) \geq V_\epsilon \text{ on } \|\vec{x}\| = \epsilon . \qquad (C.4)$$

Since V is positive definite it is clear that $V_\epsilon > 0$. Because V is continuous and $V(\vec{0}) = 0$ there exists a $\delta = \delta(\epsilon)$ such that if motion initiates at a point $\vec{x}_o = \vec{x}(t_o)$ inside a spherical domain $\|\vec{x}\| \leq \delta$, we have

$$V(\vec{x}_o) < V_\epsilon . \qquad (C.5)$$

[1]Theorem and proof taken from Reference [7], pages 234-235.

The fact that $\dot{V}$ is negative semidefinite in D_h implies that

$$V(\vec{x}(\vec{x}_o,t)) \le V(\vec{x}_o) < V_\epsilon \quad \text{for} \quad t > t_o \quad , \tag{C.6}$$

where $\vec{x}(\vec{x}_o,t)$ is the solution corresponding to the initial condition $\vec{x}_o$. From (C.6) we must conclude

$$\|\vec{x}(\vec{x}_o,t)\| \le \epsilon, \quad t > t_o \quad . \tag{C.7}$$

Now, let us assume that at a certain time $t = T$

$$\|\vec{x}(\vec{x}_o,T)\| = \epsilon \ , \tag{C.8}$$

thus, because of (C.4), we must have

$$V(\vec{x}(\vec{x}_o,T) \ge V_\epsilon \ . \tag{C.9}$$

This is a contradiction to (C.6). Therefore if the solution $\vec{x}(t)$ of Equation (C.1) is such that the inequality (C.6) is satisfied, the motion will remain in the domain $\|\vec{x}\| < \epsilon$ for all $t > t_o$, hence the equilibrium is a stable point. Q.E.D.

Theorem 2[1]. If there exists for the system (C.1) a positive (negative) definite function $V(\vec{x})$ such that $\dot{V}(\vec{x})$ is negative (positive) definite along every trajectory of (C.1), then the origin is asymptotically stable.

Proof: The proof that the origin is at least a stable point follows from Theorem 1. Thus, it remains for us to show that $\vec{x}(t) \to 0$ as $t \to +\infty$.

[1] Theorem taken from Reference [7], pages 235-236.

Let $\dot{V}$ be negative definite in D_h. Then V must be monotonically decreasing with time for all $\vec{x} \neq 0$. We assume that there is a positive number b such that

$$\dot{V}(\vec{x}) \leq -b . \tag{C.10}$$

It follows that for $t > t_o$,

$$\int_{t_o}^{t} \dot{V}(\vec{x})dt \leq -b(t - t_o) \tag{C.11}$$

or

$$V(\vec{x}) \leq V(\vec{x}_o) - b(t - t_o) . \tag{C.12}$$

But (C.12) implies that for sufficiently large t, V becomes negative in D_h which is a contradiction to the condition of the theorem that V be positive definite in D_h. Thus, we conclude that there exist no such positive number b that satisfies inequality (C.10). Therefore, V does not stall above a certain value $\alpha > 0$, but tends to zero. Hence, we conclude that

$$\lim_{t \to \infty} V(\vec{x}) = 0 , \tag{C.13}$$

from which it follows that

$$\lim_{t \to \infty} \vec{x}(t) = \vec{0} . \tag{C.14}$$

Q.E.D.

C.3 The Usual Procedure Followed for the Construction of a Lyapunov Function

The procedure chosen by most investigators in seeking a Lyapunov function $V(\vec{x})$ for a dynamical system such as (C.1), is to first obtain all of the integrals of motion $V_1, \ldots, V_m$, associated with the system, where it is clear that V_i = const., for $i = 1, \ldots, m$. Next, an attempt is made to construct a function V from the V_i's which is quadratic in the variables x_j, $j = 1, \ldots, n$, where $\vec{x}$ is an n-vector. Since the V_i's are constant, then $\dot{V} \equiv 0$. Therefore, it trivally follows that $\dot{V}$ is negative semidefinite.

The final step involves determination of the conditions which make V positive definite, which follow easily from the well-known Sylvester's Criterion.

It should be pointed out that there is no unique Lyapunov function associated with a particular system, nor is the method of obtaining such functions regimented. Finding a V that is positive definite and quadratic in its variables many times proves to be the most difficult step in the construction of the Lyapunov function. It is at this point where many investigators seek other methods to analyze the stability of their systems.

APPENDIX D

A DISCUSSION OF THE APPLICABILITY OF THE DIRECT METHOD OF LYAPUNOV TO THE EXTENDED GYROSTAT OF CHAPTER IV

In any stability analysis of a dynamical system it is most desirable to obtain sufficient conditions for stability of the full system. When the differential equations of motion are nonlinear or have time-varying coefficients, such an analysis may prove to be very cumbersome. Due to a lack of methodology, one might be required to go to great lengths to obtain a suitable Lyapunov testing function which would yield nontrivial information regarding the stability of the system under investigation.

The purpose of this appendix is to discuss a difficulty which arose in seeking to obtain a suitable Lyapunov function for the case of the extended gyrostat G^* considered in Chapter IV. Recalling the equations of motion of G^*, we write

$$(I_x - e\,S_\theta^2)\dot{\omega}_x + \frac{1}{2}\,eS_{2\theta}\,\dot{\omega}_y - \frac{1}{2}\,e\,S_{2\theta}\,\omega_x\omega_z$$

$$+ (I_z - I_y - eS_\theta^2)\omega_y\dot{\omega}_z - eS_{2\theta}\omega\omega_x + (eC_{2\theta} + I_{z2})\omega\omega_y$$

$$= mgd\,\gamma_2 \quad , \tag{D.1}$$

$$\frac{1}{2}\,e\,S_{2\theta}\,\dot{\omega}_x + (I_y + eS_\theta^2)\dot{\omega}_y + \frac{1}{2}\,e\,S_{2\theta}\,\omega_y\omega_z$$

$$+ (I_x - I_z - eS_\theta^2)\omega_x\omega_z + (eC_{2\theta} - I_{z_2})\omega\omega_x + eS_{2\theta}\omega\omega_y$$

$$= mgd\gamma_1 \quad , \tag{D.2}$$

$$I_z\dot{\omega}_z + \frac{1}{2}\, eS_{2\theta}\omega_x^2 + [I_y - I_x + e(1 - C_{2\theta})]\omega_x\omega_y - \frac{1}{2}\, eS_{2\theta}\omega_y^2 = 0 \; . \tag{D.3}$$

$$\dot{\gamma}_1 = \omega_z\gamma_2 - \omega_y\gamma_3 \;, \tag{D.4}$$

$$\dot{\gamma}_2 = \omega_x\gamma_3 - \omega_z\gamma_1 \;, \tag{D.5}$$

$$\dot{\gamma}_3 = \omega_y\gamma_1 - \omega_x\gamma_2 \;, \tag{D.6}$$

where all of the parameters are clearly defined in Chapter IV.

Now following the procedure of Rumiantsev [18], we seek the integrals of motion from which a useful Lyapunov function is ordinarily able to be constructed.

Obviously, Equations (D.4-D.6) admit the geometrical integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \;, \tag{D.7}$$

as was used by Rumiantsev [18].

Next, we multiply Equations (D.1-D.3) by γ_1, γ_2 and γ_3, respectively, and add. After some mathematical manipulation and the use of Equations (D.1-D.6), we obtain a second integral which we write as

$$\begin{aligned}(I_x - eS_\theta^2)\omega_x\gamma_1 + (I_y + eS_\theta^2)\omega_y\gamma_2 + I_z\omega_z\gamma_3 + I_{z_2}\omega\gamma_3 \\ + \frac{1}{2}\, eS_{2\theta}\omega_y\gamma_1 + \frac{1}{2}\, eS_{2\theta}\omega_x\gamma_2 \\ = \text{const.},\end{aligned} \tag{D.8}$$

which represents the conservation of angular momentum about the vertical.

If we set $e = 0$ in Equation (D.8), our equation coincides with the angular momentum integral obtained by Rumiantsev for the gyrostat in [18].

We now seek an energy integral for the extended gyrostat. If we multiply Equations (D.1-D.3) by ω_x, ω_y, and ω_z, respectively, then add and simplify, the result obtained is

$$(I_x - eS_\theta^2)\omega_x\dot{\omega}_x + (I_y + eS_\theta^2)\omega_y\dot{\omega}_y + I_z\omega_z\dot{\omega}_z + \frac{1}{2}\, eS_{2\theta}\omega_y\dot{\omega}_x$$

$$+ \frac{1}{2}\, S_{2\theta}\omega_x\dot{\omega}_y + 2eC_{2\theta}\omega\omega_x\omega_y - eS_{2\theta}\omega\omega_x^2 + eS_{2\theta}\omega\omega_y^2$$

$$= mgd(\omega_x\gamma_2 - \omega_y\gamma_1) \quad . \tag{D.9}$$

Substituting Equation (D.6) into Equation (D.9) we can easily show

$$\frac{d}{dt}\left[(I_x - eS_\theta^2)\frac{\omega_x^2}{2} + (I_y + eS_\theta^2)\frac{\omega_y^2}{2} + I_z\,\frac{\omega_z^2}{2} + \frac{1}{2}\, eS_{2\theta}\omega_x\omega_y + mgd\gamma_3\right]$$

$$+ eC_{2\theta}\omega\omega_x\omega_y - \frac{1}{2}\, eS_{2\theta}\omega\omega_x^2 + \frac{1}{2}\, eS_{2\theta}\omega\omega_y^2 = 0 \quad . \tag{D.10}$$

Clearly if $e = 0$ (the case of a gyrostat) Equation (D.10) simplifies to

$$\frac{d}{dt}\left[I_x\,\frac{\omega_x^2}{2} + I_y\,\frac{\omega_y^2}{2} + I_z\,\frac{\omega_z^2}{2} + mgd\gamma_3\right] = 0 \quad , \tag{D.11}$$

which yields the integral

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 + 2\ mgd\gamma_3 = \text{const.} \tag{D.12}$$

Equation (D.12) was the first integral obtained by Rumiantsev and represents the conservation of total energy for the system. However, if $e \neq 0$ the reduction of Equation (D.10) to an integral is not obvious.

Recalling from Chapter IV that the extended gyrostat G^* consisted of two rigid bodies B_1 and B_2, the angular momentum of body B_2 alone about its own mass center follows from the definition of angular momentum. If G_2 denotes the mass center of B_2, then the angular momentum of B_2 about G_2 is given by

$$\vec{H}_{G_2}^{B_2} = \int_{B_2} [\vec{\rho} \times ({}^{F}\vec{\omega}^{B_2} \times \vec{\rho})] dm_p \ , \tag{D.13}$$

where ${}^{F}\vec{\omega}^{B_2}$ and $\vec{\rho}$ were defined in Chapter IV as follows:

$$ {}^{F}\vec{\omega}^{B_2} = \omega_x \hat{i} + \omega_y \hat{j} + (\omega_2 + \omega)\hat{k} \ , \tag{D.14}$$

$$\vec{\rho} = (x_2 c_\theta - y_2 s_\theta)\hat{i} + (x_2 s_\theta + y_2 c_\theta)\hat{j} + z_2 \hat{k} \ . \tag{D.15}$$

Substituting Equations (D.14,D.15) into Equation (D.13) and recalling that (x_2, y_2, z_2) are the principal axes of B_2 for the point G_2 the integration of Equation (D.13) yields

$$\begin{aligned} \vec{H}_G^{B_2} &= \left[(I_{x_2} C_\theta^2 + I_{y_2} S_\theta^2)\omega_x + \frac{1}{2}\, e\, S_{2\theta}\omega_y\right]\hat{i} \\ &+ \left[\frac{1}{2}\, e\, S_{2\theta}\omega_x + (I_{y_2} S_\theta^2 + I_{y_2} C_o^2)\omega_y\right]\hat{j} + I_{z_2}(\omega_2 + \omega)\hat{k} \ . \end{aligned} \tag{D.16}$$

The third component of the moment equation of B_2 with respect to the mass center G_2 is given by

$$\hat{k} \cdot \left(\overset{B_1 \cdot B_2}{\dot{\vec{H}}_{G_2}} + \overset{F\ B_1}{\vec{\omega}} \times \overset{B_2}{\vec{H}_{G_2}} \right) = \overset{B_2}{\vec{M}_{G_2}} \cdot \hat{k} , \tag{D.17}$$

where $\overset{B_2}{\vec{M}_{G_2}}$ is the total moment exerted on body B_2 about its mass center G_2 and

$$\overset{F\ B_1}{\vec{\omega}} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \tag{D.18}$$

was defined in Chapter IV.

Now we substitute Equations (D.16, D.18) into Equation (D.17) and perform the indicated operations. This gives us

$$I_{z_2} \dot{\omega}_2 + \frac{1}{2} e\ S_{2\theta}(\omega_x^2 - \omega_y^2) - e\ C_{2\theta}\omega_x\omega_y = \overset{B_2}{\vec{M}_{G_2}} \cdot \hat{k} , \tag{D.19}$$

where $\omega = \dot{\theta} = \text{const.}$

Multiplication of Equation (D.19) by ω yields

$$I_{z_2} \omega\dot{\omega}_z + \frac{1}{2} e\ S_{2\theta}\omega(\omega_x^2 - \omega_y^2) - e\ C_{2\theta}\omega\omega_x\omega_y = \omega \overset{B_2}{\vec{M}_{G_2}} \cdot \hat{k} , \tag{D.20}$$

which can be rewritten in the more useful form

$$- \frac{1}{2} e\ S_{2\theta}\ \omega(\omega_x^2 - \omega_y^2) + e\ C_{2\theta}\omega\omega_x\omega_y = I_{z_2} \omega\dot{\omega}_z - \omega\vec{M}_{G_2} \cdot \hat{k} . \tag{D.21}$$

Hence, if we substitute Equation (D.21) into Equation (D.10) we obtain the equation

$$\frac{d}{dt}\left[(I_x - e\,S_\theta^2)\frac{\omega_x^2}{2} + (I_y + e\,S_\theta^2)\frac{\omega_y^2}{2} + I_z\frac{\omega_z^2}{2} + \frac{1}{2}\,e\,S_{2\theta}\omega_x\omega_y + mgd\gamma_3 + I_{z_2}\,\omega\omega_z\right] = \omega\,\vec{M}_{G_2}^{B_2}\cdot\hat{k}\,. \qquad (D.22)$$

Since we have initially made the assumption $\omega = const.$ (i.e. $\theta = \omega t$) it should be realized that there must be some moment $\vec{M}_{G_2}^{B_2}$ exerted on body B_2 by B_1 to maintain this condition. However, the question arises of how one specifies $\vec{M}_{G_2}^{B_2}$ so that we might obtain an integral of motion. This, of course, places great restrictions on the problem. If indeed there is a potential function U such that $\dot{U} = -\,\omega\,\vec{M}_{G_2}^{B_2}\cdot\hat{k}$, then one might become worried about the definiteness of U after the integration. Also, we would need to know how the perturbations effect U. Therefore it does not seem beneficial to seek a Lyapunov function of quadratic form from an integral containing U.

Fortunately in the case of a gyrostat this problem can easily be eliminated. Writing down the energy as the kinetic energy plus the potential energy and recalling that only conservative forces are acting on the gyrostat, we obtain

$$I_x\frac{\omega_x^2}{2} + I_y\frac{\omega_y^2}{2} + I_z\frac{\omega_z^2}{2} + I_{z_2}\,\omega\omega_z + mgd\gamma_3 + U = \text{const.}\ , \qquad (D.23)$$

where

$$\dot{U} = -\,\omega\,\vec{M}_{G_2}^{B_2}\cdot\hat{k}\,.$$

But in the case of a gyrostat e = 0 and Equation (D.21) reduces to

$$I_{z_2}\,\omega\dot{\omega}_z + \dot{U} = 0\ ; \qquad (D.24)$$

hence, after integration we obtain

$$I_{z_2} \omega\omega_z + U = \text{const.} \tag{D.25}$$

Substituting Equation (D.25) into Equation (D.23) we obtain the new energy integral

$$I_x \frac{\omega_x^2}{2} + I_y \frac{\omega_y^2}{2} + I_z \frac{\omega_z^2}{2} + mgd\gamma_3 = \text{const.} \;, \tag{D.26}$$

which no longer contains the potential function U. Equation (D.26) is seen to be nothing more than Equation (D.12) again.

Nevertheless and unfortunately, this simplification does not occur for the extended gyrostat as we have seen. Only by introducing some restrictions on U, in the case of the extended gyrostat, could we obtain a meaningful Lyapunov function. However, this would require additional constraints which would result in a loss in generality. Therefore, the stability analysis using the direct method of Lyapunov was abandoned at this point.

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VITA

Luther Street Long, III was born January 4, 1947 in Philadelphia, Mississippi. He attended the public schools of Guntersville, Alabama and graduated from Guntersville High School in 1965. He received a Bachelor of Science in mathematics from Jacksonville State University located in Jacksonville, Alabama. He entered graduate school at Georgia Tech in 1970 and received a Master of Science in Applied Mathematics in 1972 and a Master of Science in Engineering Science in 1973.